1. Exercise on 12 October 2026

[LVA 405.161 UE Statistics, tasks marked with (*) are voluntary and have an increased level of difficulty]

Exercise 1 (Warm-up exercise) Prove the following two statements for sequences of sets

$$\liminf_{n\to\infty} A_n = \{x \in \Omega : \exists n_0 \in \mathbb{N} \text{ with } x \in A_n \text{ for every } n \geq n_0\}$$
$$\limsup_{n\to\infty} A_n = \{x \in \Omega : x \in A_n \text{ for infinitely many } n \in \mathbb{N}\}.$$

Exercise 2 (Warm-up exercise) The indicator function $1_A : \Omega \to \{0, 1\}$ of a set $A \subseteq \Omega$ is known to be defined by

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in A^c. \end{cases}$$

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of arbitrary sets, $\underline{A} \coloneqq \liminf_{n\to\infty} A_n$ and $\overline{A} \coloneqq \limsup_{n\to\infty} A_n$. Do the following equations hold for every $x \in \Omega$?

$$1_{\underline{A}}(x) = \liminf_{n\to\infty} 1_{A_n}(x), \qquad 1_{\overline{A}}(x) = \limsup_{n\to\infty} 1_{A_n}(x).$$

Hint: Use Exercise 1.

Exercise 3 We construct the sequence of sets $(A_n)_{n\in\mathbb{N}}$ as follows (repeated removal of the middle third, a sketch is helpful):

$$A_1 \coloneqq [0, 1], A_2 \coloneqq A_1 \setminus \left(\tfrac{1}{3}, \tfrac{2}{3}\right), A_3 \coloneqq A_2 \setminus \left(\left(\tfrac{1}{9}, \tfrac{2}{9}\right) \cup \left(\tfrac{7}{9}, \tfrac{8}{9}\right)\right), \ldots$$

Calculate $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$. How many points does $\liminf_{n\to\infty} A_n$ contain?

Exercise 4 Determine $\sigma(\mathcal{E})$ for the following generators $\mathcal{E}$ and sample spaces Ω:

a) $\Omega = \{1, 2, 3\}, \mathcal{E} = \big\{\{1, 3\}\big\}$

b) $\Omega = \mathbb{Q} = \{q_1, q_2, \ldots\}, \mathcal{E} = \big\{\{q_i\} : i \geq k\big\}$ for a fixed $k \in \mathbb{N}$

c) $\Omega = \mathbb{R}, \mathcal{E} = \big\{(-\infty, a] : a \in \mathbb{R}\big\}$

Exercise 5 Show, for at least two of the 10 remaining set systems in Lemma 1.14, that they are generators of the Borel σ-algebra. Use the proofs for $\mathcal{E}_1$ and $\mathcal{E}_2$ contained in the lecture notes as a guide.

Exercise 6 Let $q_1, q_2, \ldots$ be an enumeration of the rational numbers in $[0,1]$ and $r_n = \frac{1}{2}\frac{1}{4^n}$ for every $n \in \mathbb{N}$. Furthermore, let the set U be defined by

$$U := \bigcup_{n=1}^{\infty} [q_n - r_n, q_n + r_n].$$

Since the rational numbers in $[0,1]$ are dense, one might assume that $U = [0,1]$. Is this statement correct? **Hint:** Estimate $\lambda(U)$.