Baire category results for quasi–copulas*

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Abstract

The aim of this manuscript is to determine the relative size of several functions (copulas, quasi–copulas) that are commonly used in stochastic modeling. It is shown that the class of all quasi–copulas that are (locally) associated to a doubly stochastic signed measure is a set of first category in the class of all quasi–copulas. Moreover, it is proved that copulas are nowhere dense in the class of quasi-copulas. The results are obtained via a checkerboard approximation of quasi–copulas.

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1 Introduction

The concept of a quasi–copula was introduced by Alsina et al. in [1] in order to characterize those operations on random variables that can be derived from operations on the related distribution functions (see also [5; 14; 20]). Since then, it also has appeared in various problems related to stochastic dependence [4], multicriteria decision making [15; 16; 17; 19] and probabilistic metric spaces [27; 28]. For more details, see [6, chapter 7] and the references therein.

It is well known that the class of quasi–copulas includes the class of copulas [3; 6], which are nowadays standard tools for dealing with a variety of problems related to stochastic models. From a measure–theoretic viewpoint, however, these two objects differ in a significant way. While copulas are in one-to-one correspondence with (σ–additive) probability measures on $[0,1]^2$, quasi–copulas may, in general, neither be associated to positive measures (since they can assign negative mass to specific rectangles in $[0,1]^2$) nor to signed measures, as recently proved by [9; 10; 23].

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The aim of this note is to clarify the relationships among several sub-classes of quasi–copulas that are characterized in terms of properties of their induced measures. In particular, we will describe the relative size of these classes in the language of Baire categories (see, e.g. [24]). In order to prove our main results, we will define a powerful methods to approximate quasi–copulas by means of special matrices.

We would like to stress that the use of Baire categories to characterize special classes of copulas (and related functions) originated from an open problem proposed by [2, Problem 10] (see also [8]), and has been recently applied to various contexts by [7].

2 Quasi-transformation matrices and checkerboard approximations

In the sequel $\mathscr{Q}$ denotes the class of all two-dimensional quasi–copulas, $\mathscr{C}$ the family of all two-dimensional copulas, Π denotes the independence copula and M the Fréchet–Hoeffding upper bound copula. All elements of $\mathscr{Q} \setminus \mathscr{C}$ are called *proper* quasi–copulas. Letting d_∞ denote the uniform metric on $\mathscr{Q}$, Ascoli–Arzelá theorem implies compactness of $(\mathscr{Q}, d_\infty)$ as well as of $(\mathscr{C}, d_\infty)$. For further properties of copulas and quasi-copulas we refer to [6; 21].

It is well-known that every copula $A \in \mathscr{C}$ corresponds to a unique doubly stochastic measure μ_A on the Borel σ-field $\mathscr{B}([0,1]^2)$ but not all quasi-copulas correspond to a doubly stochastic signed measure on $\mathscr{B}([0,1]^2)$, i.e. there are quasi-copulas $Q \in \mathscr{Q}$ for which there exists no (doubly stochastic) signed measure μ (see [26; 18]) on $\mathscr{B}([0,1]^2)$ fulfilling

$$\mu([\underline{a},\overline{a}] \times [\underline{b},\overline{b}]) = Q(\overline{a},\overline{b}) - Q(\overline{a},\underline{b}) - Q(\underline{a},\overline{b}) + Q(\underline{a},\underline{b}) := V_Q([\underline{a},\overline{a}] \times [\underline{b},\overline{b}]) \tag{2.1}$$

for all rectangles $[\underline{a},\overline{a}] \times [\underline{b},\overline{b}] \subseteq [0,1]^2$. As usual we will refer to $V_Q(R)$ as Q-volume of the compact rectangle R. In the sequel $\mathscr{Q}_S$ will denote the family of all quasi-copulas Q for which there exists a doubly stochastic signed measure μ_Q fulfilling eq. (2.1) for all rectangles $[\underline{a},\overline{a}] \times [\underline{b},\overline{b}] \subseteq [0,1]^2$. One possible construction of elements in $\mathscr{Q}_S^c$ is via so-called quasi-transformation matrices (which are natural generalizations of transformation matrices as studied in [13; 11]) and the induced Iterated Functions Systems. We will first recall the construction going back to [9], discuss basic properties and as well as some examples that will be used later on, and then concentrate on checkerboard approximations which are, in fact, strongly related.

A matrix $\tau = (t_{ij})_{i,j=1}^m \in [-1,1]^{m\times m}$ with $m \geq 2$ is called a *quasi-transformation matrix* if and only if the following three conditions hold:

1. $\sum_{i,j=1}^m t_{ij} = 1$
2. $\sum_{j=1}^m t_{ij} > 0$ for every $i \in \{1,\dots,m\}$ as well as $\sum_{i=1}^m t_{ij} > 0$ for every $j \in \{1,\dots,m\}$
3. $\sum_{i:\underline{i}\leq i\leq \overline{i}} \sum_{j:\underline{j}\leq j\leq \overline{j}} t_{ij} > 0$ whenever $\underline{i} \in \{1,m\}$, or $\overline{i} \in \{1,m\}$, or $\underline{j} \in \{1,m\}$ or $\overline{j} \in \{1,m\}$.

Notice that these three conditions (using arguments similar to the proof of Theorem 7.4.3 in [6]) are easily seen to imply $\tau \in [-\frac{1}{3}, 1]^{m \times m}$. In the sequel $\mathscr{T}$ will denote the family of all quasi-transformation matrices. $\tau \in \mathscr{T}$ will be called *proper* if at least one entry is strictly negative.

Given a quasi-transformation matrix $\tau \in [-1,1]^{m\times m}$, let p_i (respectively, q_j) denote the sum of the entries in the first i columns (respectively, first j rows) of τ, for every i (respectively, j) in $\{1,\dots,m\}$ and set $p_0 = q_0 = 0$. Having this and defining $R_{ij} := [p_{i-1}, p_i] \times [q_{j-1}, q_j]$ for all $i,j \in \{1,\dots,m\}$ yields a family of m^2 non-empty compact rectangles with pairwise disjoint interior whose union is $[0,1]^2$. Every transformation matrix $\tau \in [0,1]^{m\times m}$ induces an operator $\mathscr{W}_\tau$ on the family $\mathscr{Q}$ of all two-dimensional quasi-copulas. In fact, considering $(x,y) \in R_{i_0,j_0}$ and setting (empty sums are zero by definition)

$$\mathscr{W}_\tau(B)(x,y) = \sum_{i<i_0, j<j_0} t_{ij} + \tfrac{x-p_{i_0-1}}{p_{i_0}-p_{i_0-1}} \sum_{j<j_0} t_{i_0 j} + \tfrac{y-q_{j_0-1}}{q_{j_0}-q_{j_0-1}} \sum_{i<i_0} t_{i j_0} + t_{i_0 j_0} B\Big(\tfrac{x-p_{i_0-1}}{p_{i_0}-p_{i_0-1}}, \tfrac{y-q_{j_0-1}}{q_{j_0}-q_{j_0-1}}\Big), \tag{2.2}$$

it is straightforward to verify that $\mathscr{W}_\tau(B) \in \mathscr{Q}$ for every $B \in \mathscr{Q}$ and that $\mathscr{W}_\tau(B) \in \mathscr{Q} \setminus \mathscr{C}$ for proper τ and arbitrary $B \in \mathscr{Q}$. Furthermore (see again [9]) it can be shown that $\mathscr{W}_\tau$ is a contraction on the compact metric space $(\mathscr{Q}, d_\infty)$ whenever $\tau \in \mathscr{T}$ fulfills $L := \max\{|t_{ij}| : 1 \le i,j \le m\} < 1$. In the latter case, Banach's fixed point theorem implies the existence of a unique, globally attractive fixed point $B^\star_\tau \in \mathscr{Q}$. Whenever τ is proper, according to [9] the quasi-copula $B^\star_\tau$ has no corresponding doubly stochastic signed measure, i.e. $B^\star_\tau \notin \mathscr{Q}_S$. In fact, if $\mu_n^+ - \mu_n^-$ denotes the Hahn decomposition (see [26]) of the signed measure induced by $\mathscr{W}_\tau^n(\Pi) \in \mathscr{D}_S$, then both sequences $(\mu_n^+([0,1]^2)_{n\in\mathbb{N}}$ and $(\mu_n^-([0,1]^2)_{n\in\mathbb{N}}$ are strictly increasing and unbounded whenever τ is proper.
One obvious, but essential property of the operator $\mathscr{W}_\tau$ that we will use in the sequel is the fact that $\mathscr{W}_\tau(\mathscr{Q}_S^c) \subseteq \mathscr{Q}_S^c$ as well as $\mathscr{W}_\tau(\mathscr{Q}_S) \subseteq \mathscr{Q}_S$. Additionally, letting $\mathscr{Q}_{ac}$ the family of all elements on $\mathscr{Q}_S$ for which the corresponding doubly stochastic signed measure is absolutely continuous with bounded density, we also have $\mathscr{W}_\tau(\mathscr{Q}_{ac}) \subseteq \mathscr{Q}_{ac}$.

Remark 2.1. As in the case of (non-negative) transformation matrices (see [13; 11]) it seems natural to extend the notion of a quasi-transformation matrix to arbitrary dimensions $d \geq 3$. The exact conditions which such a d-dimensional quasi-transformation matrix should fulfill, however, are unclear since the property that the Q-volume V_Q fulfills that $V_Q(R) \geq 0$ for all d-dimensional rectangles R with at least $d-1$ adjacent faces being contained in the boundary of $[0,1]^d$ together with the standard boundary conditions is known to be sufficient but not necessary for Q to be a d-dimensional quasi-copula (see [25]).

Example 2.1. We consider the proper quasi-transformation matrix $\tau \in [-1,1]^{3\times 3}$ given by

$$\tau_1 = \begin{pmatrix} 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 \end{pmatrix}.$$

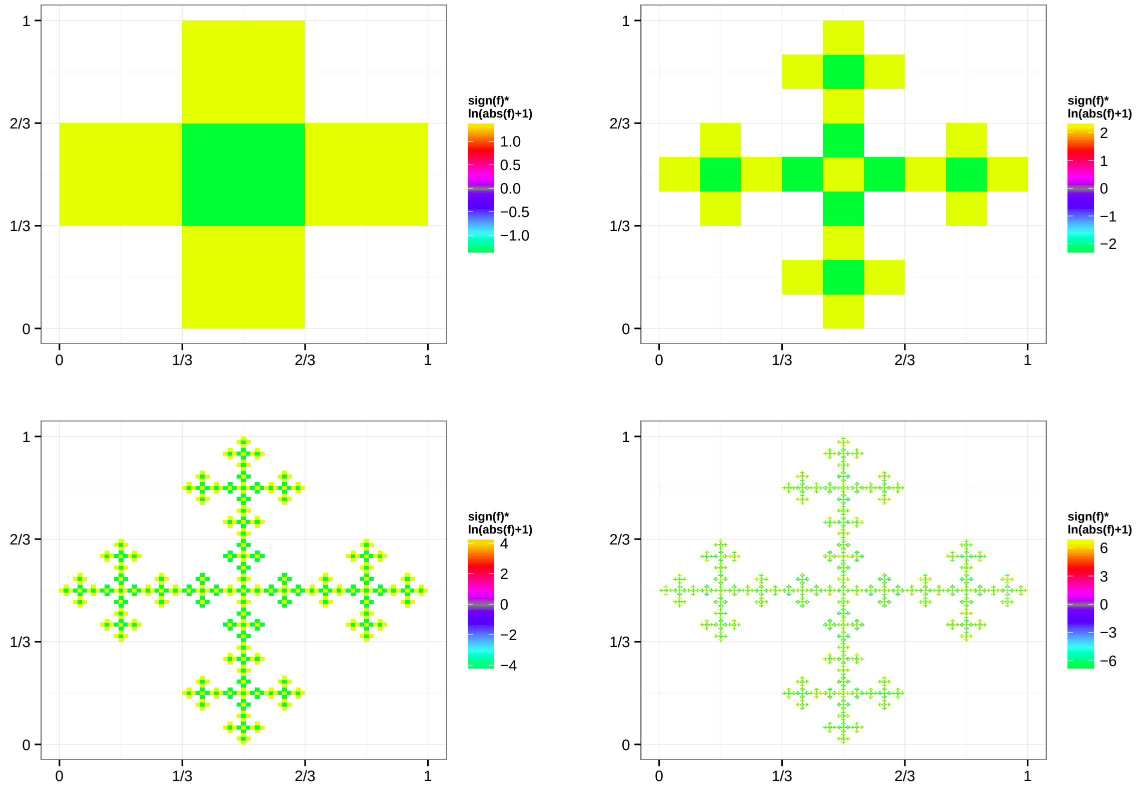

Figure 1: Density f of $\mathscr{W}_{\tau_1}^n(\Pi)$, with τ_1 according to Example 2.1 and $n \in \{1,2,4,6\}$

Figure 1 depicts the (signed) density f of the doubly stochastic signed measure corresponding to the proper quasi-copula $\mathscr{W}_{\tau_1}^n(\Pi)$, whereby $n \in \{1,2,4,6\}$.

Example 2.2. We consider the proper quasi-transformation matrix $\tau_2 \in [-1,1]^{3\times 3}$ given by

$$\tau_2 = \tfrac{1}{18}\begin{pmatrix} 1 & 4 & 1 \\ 4 & -2 & 4 \\ 1 & 4 & 1 \end{pmatrix}.$$

Figure 2 depicts the (signed) density f of the doubly stochastic signed measure corresponding to the proper quasi-copula $\mathscr{W}_{\tau_2}^n(\Pi)$, whereby this time $n \in \{1,2,4,5\}$.

Based on the results mentioned before, none of the two proper quasi-copulas $B_{\tau_1}^\star$ and $B_{\tau_2}^\star$ with τ_1, τ_2 as in Example 2.1 and Example 2.2 induces a doubly stochastic signed measure. There is, however, a fundamental difference between $B_{\tau_1}^\star$ and $B_{\tau_2}^\star$: The latter does not allow an extension of the $B_{\tau_2}^\star$-volume $V_{B_{\tau_2}^\star}$ to a signed measure on $\mathscr{B}([0,1]^2) \cap B((x,y),r)$ for any choice of $(x,y) \in [0,1]^2$ and $r > 0$, whereby $B((x,y),r)$ denotes the open ball of radius r around (x,y). Letting λ_2 denote the Lebesgue measure on $[0,1]^2$, the former, on

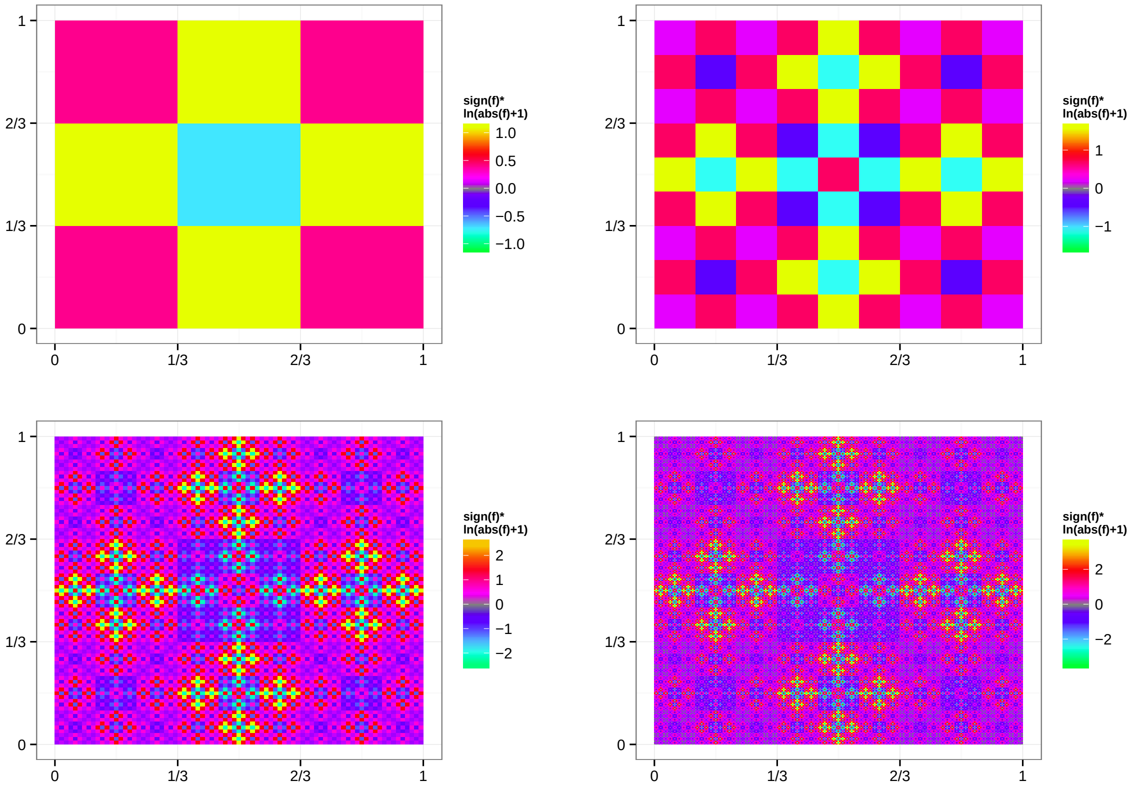

Figure 2: Density f of $\mathscr{W}^n_{\tau_2}(\Pi)$, with τ_2 according to Example 2.2 and $n \in \{1, 2, 4, 6\}$

the contrary, allows to extend $V_{B^\star_{\tau_1}}$ locally at λ_2-almost every point to a signed measure, namely the trivial measure assigning each Borel set measure zero.

Definition 2.1. A quasi-copula Q is called *locally extendable* if there exists a point $(x, y) \in [0,1]^2$ and some $r > 0$ such that V_Q can be extended to a signed measure on $\mathscr{B}([0,1]^2) \cap B((x,y), r)$. In the sequel $\mathscr{Q}^{loc}_S$ will denote the class of all locally extendable quasi-copulas.

We will see in the next section that (topologically speaking) typical quasi-copulas are not even locally extendable.

Suppose now that $Q \in \mathscr{Q}$, fix $m \geq 1$ and set

$$t^Q_{ij} := V_Q\big([\tfrac{i-1}{2^m}, \tfrac{i}{2^m}] \times [\tfrac{j-1}{2^m}, \tfrac{j}{2^m}]\big) \tag{2.3}$$

for all $i, j \in \{1, \ldots, 2^m\}$. Then $\tau^Q := (t^Q_{ij})^{2^m}_{i,j=1}$ is a $2^m \times 2^m$ quasi-transformation matrix inducing the operator $\mathscr{W}^Q_m := \mathscr{W}_{\tau^Q}$ on $\mathscr{Q}$.

Definition 2.2. Given quasi-copulas Q and B, the quasi-copula $\mathscr{W}^Q_m(B)$ is called the *B-checkerboard (quasi-copula) of Q of order m*. In case of $B = \Pi$ we will refer to $\mathscr{W}^Q_m(B)$ simply as *checkerboard of Q of order m*.

Example 2.3. We consider checkerboards of A_δ with A_δ being the maximal quasi-copula (MQC, for short) with given diagonal δ as studied in [22] and [29]. The MQC A_δ is given by

$$A_\delta(x,y) := \min\Big\{x, y, \max\{x,y\} - \max\Big\{t - \delta(t) : t \in \big[\min\{x,y\}, \max\{x,y\}\big]\Big\}\Big\} \tag{2.4}$$

for all $x, y \in [0,1]$. According to [12] $A_\delta \in \mathscr{Q}_S$ and the positive as well as the negative part of the Hahn decomposition of A_δ are singular and concentrate their mass on the graphs of (at most) three functions. Choosing δ as in the left part of Figure 3 A_δ has the form shown in Figure 4. The right part of Figure 3 depicts the (signed) density of the checkerboard $\mathscr{W}_4^{A_\delta}(\Pi)$.

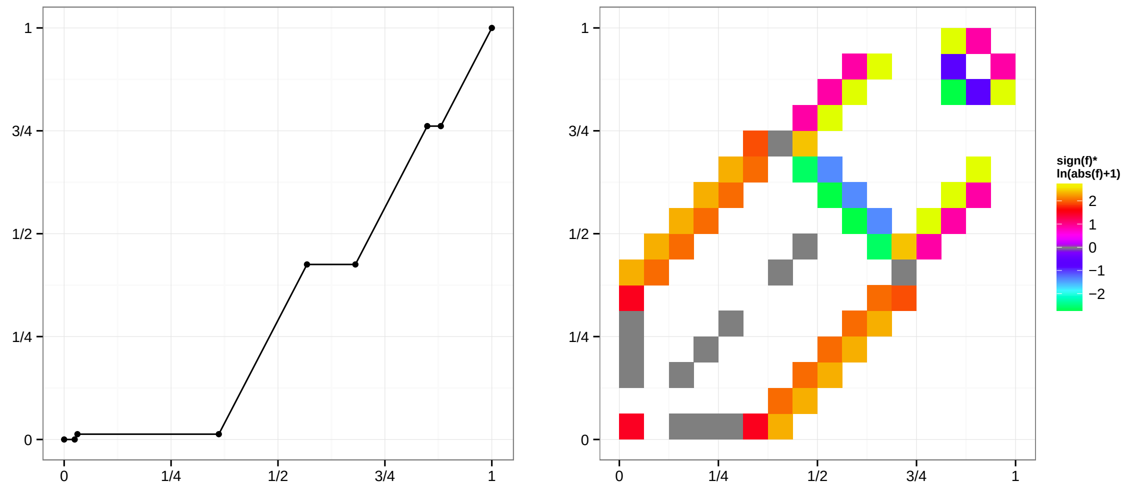

Figure 3: Diagonal δ (left) and density f (right) of $\mathscr{W}_4^{A_\delta}(\Pi)$, with A_δ according eq. (2.4)

Considering that in case of $m = 1$ we obviously have $0 \leq t_{ij}^Q \leq \frac{1}{2}$ and that in case of $m \geq 2$ Lipschitz continuity implies $|t_{ij}^Q| \leq \frac{1}{2^m} < 1$ for every $Q \in \mathscr{Q}$ and all $i, j \in \{1, \ldots, m\}$ it follows that $\mathscr{W}_m^Q$ is a contraction on $(\mathscr{Q}, d_\infty)$. More importantly, the checkerboard $\mathscr{W}_m^Q(B)$ of B of order m converges to Q as $m \to \infty$ for every choice of quasi-copula $B \in \mathscr{Q}$:

Theorem 2.1. *Suppose that Q and B are quasi-copulas. Then we have*

$$\lim_{m\to\infty} d_\infty\big(\mathscr{W}_m^Q(B), Q\big) = 0. \tag{2.5}$$

Proof. It follows directly from the construction of $\mathscr{W}_m^Q(B)$ that $\mathscr{W}_m^Q(B)(g_x, g_y) = Q(g_x, g_y)$ holds for every $(g_x, g_y) \in \{0, \frac{1}{2^m}, \ldots, \frac{2^m-1}{2^m}, 1\}^2 =: G$. Considering that for every $(x, y) \in$

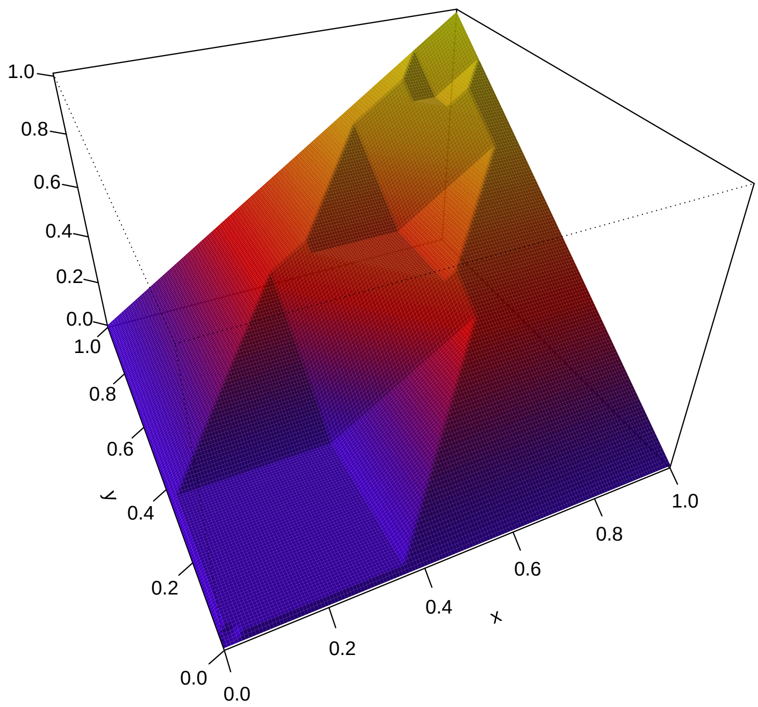

Figure 4: The (proper) quasi-copula A_δ for δ as in Figure 3.

$[0,1]^2$ there exists a point $(g_x, g_y) \in G$ with $|x - g_x| + |y - g_y| \leq \frac{1}{2^m}$, using Lipschitz continuity of quasi-copulas and the triangle inequality we get

$$\begin{aligned} |\mathscr{W}_m^Q(B)(x,y) - Q(x,y)| &\leq |\mathscr{W}_m^Q(B)(x,y) - \mathscr{W}_m^Q(B)(g_x,g_y)| + |\mathscr{W}_m^Q(B)(g_x,g_y) - Q(g_x,g_y)| \\ &\quad + |Q(g_x,g_y) - Q(x,y)| \leq \tfrac{2}{2^m} = \tfrac{1}{2^{m-1}}. \end{aligned}$$

Since $(x,y) \in [0,1]^2$ was arbitrary this directly yields $d_\infty(\mathscr{W}_m^Q(B), Q) \leq \frac{1}{2^{m-1}}$, from which the result follows immediately. □

Since $\mathscr{W}_m^Q$ maps $\mathscr{Q}_S^c$ into $\mathscr{Q}_S^c$, $\mathscr{Q}_S$ into $\mathscr{Q}_S$, and $\mathscr{Q}_{ac}$ into $\mathscr{Q}_{ac}$ for every $m \in \mathbb{N}$, Theorem 2.1 has the following direct consequence:

Corollary 2.1. *$\mathscr{Q}_S^c$, $\mathscr{Q}_S$ and $\mathscr{Q}_{ac}$ are dense in $(\mathscr{Q}, d_\infty)$.*

In the next section we will prove a much stronger result and show that $\mathscr{Q}_S^c$ is even co-meager in $(\mathscr{Q}, d_\infty)$.

3 Category results for some subsets of $(\mathscr{Q}, d_\infty)$

We recall that a subset N of a (complete) metric space (Ω, d) is called *nowhere dense* if it is not dense in any open ball $B(x, r)$ of radius $r > 0$ (equivalently, if its closure has empty interior). A set $A \subseteq \Omega$ is called *meager* or of *first category* in (Ω, d) if it can be expressed as (or covered by) a countable union of nowhere dense sets. A is called of *second category* if it is not meager. Finally, A is called *co-meager* (or residual) if $A^c = \Omega \setminus A$ is meager. Loosely speaking, we will also refer to the elements of a co-meager set as typical and to the elements of a meager set as atypical in Ω.

A typical quasi-copula is proper - the following even stronger result holds:

Theorem 3.1. *$\mathscr{C}$ is nowhere dense in $(\mathscr{Q}, d_\infty)$.*

Proof. For every $C \in \mathscr{C}$ there exists a sequence $(Q_n)_{n\in\mathbb{N}}$ of proper quasi–copulas that converges to C with respect to d_∞. In fact, setting $B_n := (1 - \frac{1}{n})C + \frac{1}{n}Q_n$ with $Q_n \in \mathscr{Q}_S$ being a proper quasi–copula for which the corresponding doubly stochastic signed measure μ_{Q_n} fulfills $\mu_{Q_n}(\bigcup_{m=1}^{M} R_m) < -2n$ for some rectangles $R_1, \ldots, R_M \subseteq [0, 1]^2$ directly yields $B_n \in \mathscr{Q} \setminus \mathscr{C}$. Notice that such copulas Q_n can be constructed by considering $\mathscr{W}^l_{\tau_1}(\Pi)$ for sufficiently large $l \in \mathbb{N}$. As immediate consequence (the closed set) $\mathscr{C}$ cannot contain any nonempty open subset of $(\mathscr{Q}, d_\infty)$. ☐

The next theorem shows that typical quasi-copulas cannot be associated with doubly stochastic signed measures on $\mathscr{B}([0, 1]^2)$:

Theorem 3.2. *$\mathscr{Q}_S$ is of first category in $(\mathscr{Q}, d_\infty)$, i.e. $\mathscr{Q}_S^c$ is co-meager in $(\mathscr{Q}, d_\infty)$.*

Proof. For every $M \in \mathbb{N}$ consider the set $\mathscr{Q}_S^M$ of all quasi-copulas $Q \in \mathscr{Q}_S$ fulfilling that for every $k \in \mathbb{N}$ and every finite collection $(R_i)_{i=1}^k$ of compact rectangles with pairwise disjoint interior

$$\sum_{i=1}^{k} \mu_Q(R_i) \in [-M + 1, M] \tag{3.1}$$

holds. Considering $\mu_Q(R_i) = V_Q(R_i)$ it follows immediately that for every sequence $(Q_n)_{\in\mathbb{N}}$ in $\mathscr{Q}_S^M$ converging uniformly to some $Q \in \mathscr{Q}$ we have $Q \in \mathscr{Q}_S^M$, i.e. $\mathscr{Q}_S^M$ is closed. Using denseness of $\mathscr{Q}_S^c$ in $(\mathscr{Q}_S, d_\infty)$ (Corollary 2.1) shows that $\mathscr{Q}_S^M$ is nowhere dense so $\bigcup_{M=1}^{\infty} \mathscr{Q}_S^M$ is of first category in $(\mathscr{Q}, d_\infty)$. Finally, since for every $Q \in \mathscr{Q}_S$ the corresponding doubly stochastic signed measure μ_Q is finite (in fact we must have $\mu([0, 1]^2) = 1$) we get $\mathscr{Q}_S \subseteq \bigcup_{M=1}^{\infty} \mathscr{Q}_S^M$, implying that $\mathscr{Q}_S$ is of first category too. ☐

We now concentrate on the class $\mathscr{Q}_S^{loc}$ which is strictly greater than $\mathscr{Q}_S$. Nevertheless even $\mathscr{Q}_S^{loc}$ is topologically very small, implying that (topologically speaking) typical quasi-copulas do not even locally induce signed measures and underlining the fact that there is only a very weak connection between quasi-copulas and doubly stochastic signed measures.

Theorem 3.3. *$\mathscr{Q}_S^{loc}$ is of first category in $(\mathscr{Q}, d_\infty)$, i.e. $\mathscr{Q} \setminus \mathscr{Q}_S^{loc}$ is co-meager in $(\mathscr{Q}, d_\infty)$.*

Proof. First, we introduce some notations. For every $m \in \mathbb{N}$ and $i, j \in \{1, \dots, 2^m\}$ define the square S_{ij}^m by $S_{ij}^m = [\frac{i-1}{2^m}, \frac{i}{2^m}] \times [\frac{j-1}{2^m}, \frac{j}{2^m}]$. Furthermore, for every $M \in \mathbb{N}$ let $\mathscr{Q}_{S_{ij}^m}^M$ denote the family of all quasi-copulas Q fulfilling that, for every $L \in \mathbb{N}$ and every finite collection $(R_l)_{l=1}^L$ of compact rectangles with pairwise disjoint interior and $\bigcup_{l=1}^L R_l \subseteq S_{ij}^m$, the property

$$\sum_{l=1}^{L} V_Q(R_l) \in [-M+1, M] \tag{3.2}$$

holds. It is straightforward to verify that $\mathscr{Q}_{S_{ij}^m}^M$ is closed. Suppose now that $Q \in \mathscr{Q}_{S_{ij}^m}^M$ for some $m, M \in \mathbb{N}$ and $i, j \in \{1, \dots, 2^m\}$, let $\varepsilon > 0$. We distinguish the following two cases:
(i) If $V_Q(S_{ij}^m) \neq 0$, then, letting B denote an arbitrary element in $\mathscr{Q}_S^c$, we can choose a sufficiently large $k \in \mathbb{N}$ so that $d_\infty(\mathscr{W}_k^Q(B), Q) < \varepsilon$ holds.
(ii) If $V_Q(S_{ij}^m) = 0$, then we set $Q_n := (1 - \frac{1}{n})Q + \frac{1}{n}\Pi$ for every $n \in \mathbb{N}$. Choose $n_0 \in \mathbb{N}$ so that $d_\infty(Q_{n_0}, Q) < \frac{\varepsilon}{2}$ and a sufficiently large $k \in \mathbb{N}$ so that $d_\infty(\mathscr{W}_k^{Q_{n_0}}(B), Q_{n_0}) < \frac{\varepsilon}{2}$ holds. The triangle inequality implies $d_\infty(\mathscr{W}_k^{Q_{n_0}}(B), Q) < \varepsilon$.
Thus, in both situations there are quasi-copulas not contained in $\mathscr{Q}_{S_{ij}^m}^M$ having distance less than ε to Q. Since $Q \in \mathscr{Q}_{S_{ij}^m}^M$ and $\varepsilon > 0$ were arbitrary, it follows that $\mathscr{Q}_{S_{ij}^m}^M$ cannot contain any open ball with positive radius, i.e. $\mathscr{Q}_{S_{ij}^m}^M$ is nowhere dense. If V_Q is locally extendable, then it is extendable to a signed measure μ on some square S_{ij}^m. Considering $V_Q(S_{ij}^m) \in [-\frac{1}{3}, 1]$, the signed measure μ has to be finite and we get

$$\mathscr{Q}_S^{loc} \subseteq \bigcup_{M=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{i,j=1}^{2^m} \mathscr{Q}_{S_{ij}^m}^M,$$

so $\mathscr{Q}_S^{loc}$ is of first category. □

4 Conclusions

Inspired by an open problem related to the study of Baire category for copulas [2], we study the category of special subsets of quasi–copulas. Interestingly, the class of quasi–copulas that are (locally) associated with a doubly stochastic signed measure is a set of first category in the class of all quasi–copulas. In other words, a typical copula cannot be associated (even locally) with a signed measure. The results are proved by using a special approximation of quasi–copulas, a tool of general interest in copula theory. Although the results are formulated in the two–dimensional case, they can be extended to higher dimensions.

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