

# Maximum asymmetry of copulas revisited

Noppadon Kamnitui\*      Juan Fernández-Sánchez†  
Wolfgang Trutschnig\*

January 30, 2018

## Abstract

Motivated by the nice characterization of copulas  $A$  for which  $d_\infty(A, A^t)$  is maximal as established independently by Nelsen [10] and Klement & Mesiar [7], we study maximum asymmetry with respect to the conditioning-based metric  $D_1$  going back to Trutschnig [11]. Despite the fact that  $D_1(A, A^t)$  is generally not straightforward to calculate, it is possible to provide both, a characterization and a handy representation of all copulas  $A$  maximizing  $D_1(A, A^t)$ . This representation is then used to prove the existence of copulas with full support maximizing  $D_1(A, A^t)$ . A comparison of  $D_1$ - and  $d_\infty$ -asymmetry including some surprising examples rounds off the paper.

## 1 Introduction

A pair  $(X, Y)$  of random variables is called exchangeable (or symmetric), if  $(X, Y)$  and  $(Y, X)$  have the same distribution. Obviously exchangeable random variables are necessarily identically distributed but not vice versa. Trying to quantify asymmetry of identically distributed (continuous) random variables, copulas naturally come into play. In fact, if  $X, Y$  have both distribution function  $F$  and  $F$  is continuous, by Sklar's theorem  $(X, Y)$  is exchangeable if, and only if, the copula  $A$  coincides with its transpose (see Section 2). Independently of each other, Nelsen [10] as well as Klement & Mesiar [7] proved in 2006 that  $d_\infty(A, A^t) \leq \frac{1}{3}$  with equality if, and only if,  $A(\frac{2}{3}, \frac{1}{3}) = \frac{1}{3}$  and  $A(\frac{1}{3}, \frac{2}{3}) = 0$  or  $A^t(\frac{2}{3}, \frac{1}{3}) = \frac{1}{3}$  and  $A^t(\frac{1}{3}, \frac{2}{3}) = 0$  holds ( $A^t$  as usual denoting the transpose, defined by  $A^t(x, y) = A(y, x)$ ). As direct consequence, a copula  $A$  has maximal  $d_\infty$ -asymmetry if, and only if,

$$\mu_A([0, \frac{1}{3}] \times [\frac{1}{3}, \frac{2}{3}]) = \mu_A([\frac{1}{3}, \frac{2}{3}] \times [\frac{2}{3}, 1]) = \mu_A([\frac{2}{3}, 1] \times [0, \frac{1}{3}]) = \frac{1}{3}$$

or

$$\mu_{A^t}([0, \frac{1}{3}] \times [\frac{1}{3}, \frac{2}{3}]) = \mu_{A^t}([\frac{1}{3}, \frac{2}{3}] \times [\frac{2}{3}, 1]) = \mu_{A^t}([\frac{2}{3}, 1] \times [0, \frac{1}{3}]) = \frac{1}{3}$$

is fulfilled, implying that the area of the support  $\text{supp}(A)$  of each such copula  $A$  is at most  $\frac{1}{3}$ .

---

\*Department for Mathematics, University of Salzburg, Salzburg, Austria, E-mails: noppadon.kam@gmail.com, wolfgang@trutschnig.net

†Grupo de Investigación de Análisis Matemático, Universidad de Almería, La Cañada de San Urbano, Almería, Spain, E-mail: juanfernandez@ual.es

In the current paper we replace  $d_\infty$  by the metric  $D_1$  on the family of all two-dimensional copulas  $\mathcal{C}$  as introduced in [11] and study analogous questions. Working with (mutually) completely dependent copulas we first show  $\max_{A \in \mathcal{C}} D_1(A, A^t) = \frac{1}{2}$  and then provide a surprisingly simple and handy representation of all copulas  $A$  fulfilling  $D_1(A, A^t) = \frac{1}{2}$ . Based on this representation we then provide examples showing that, contrary to the case of maximal  $d_\infty$ -asymmetry, copulas with maximal  $D_1$ -asymmetry can have full support, and study the interrelation of maximal  $d_\infty$ - and  $D_1$ -asymmetry. In particular, we prove that for each  $A$  with maximal  $D_1$ -asymmetry we have  $d_\infty(A, A^t) \in [0, \frac{1}{4}]$  and that for each  $A$  with maximal  $d_\infty$ -asymmetry we have  $D_1(A, A^t) = \frac{4}{9}$ .

The rest of the paper is organized as follows: Section 2 gathers some preliminaries and notations that will be used throughout the paper. Section 3 first solves a minimization problem in the class of all completely dependent copulas and provides a characterization of all mutually completely dependent copulas with maximal  $D_1$ -asymmetry. The aforementioned general handy characterization/representation of all copulas with maximal  $D_1$ -asymmetry is presented in Section 4. Section 5 contains the comparison of the two notions quantifying asymmetry and the Appendix complements some tedious calculations concerning the minimization problem in Section 3.

## 2 Notation and Preliminaries

For every metric space  $(\Omega, d)$  the Borel  $\sigma$ -field on  $\Omega$  will be denoted by  $\mathcal{B}(\Omega)$ ,  $\delta_x$  will denote the Dirac measure (concentrated) at  $x \in \Omega$ .  $\lambda$  and  $\lambda_2$  will denote the Lebesgue measure on  $\mathcal{B}(\mathbb{R})$  and  $\mathcal{B}(\mathbb{R}^2)$  respectively. For every probability measure  $\nu$  on  $\mathcal{B}(\Omega)$  the support of  $\nu$ , i.e. the complement of the union of all open sets  $U$  fulfilling  $\nu(U) = 0$ , will be denoted by  $\text{supp}(\nu)$ .

In the sequel  $\mathcal{C}$  will denote the family of all two-dimensional *copulas*,  $\mathcal{P}_{\mathcal{C}}$  the family of all *doubly stochastic measures*, i.e. the family of all probability measures on  $\mathcal{B}([0, 1]^2)$  whose marginals are uniformly distributed on  $[0, 1]$ . For every  $C \in \mathcal{C}$  the corresponding doubly stochastic measure will be denoted by  $\mu_C$ ,  $d_\infty$  will denote the uniform metric on  $\mathcal{C}$ , given by

$$d_\infty(A, B) = \sup_{(x,y) \in [0,1]^2} |A(x, y) - B(x, y)|.$$

As usual,  $M$  denotes the minimum copula,  $\Pi$  the product copula,  $W$  the lower Fréchet-Hoeffding bound, and  $A^t$  the transpose of  $A \in \mathcal{C}$ , defined by  $A^t(x, y) = A(y, x)$ . For every  $A \in \mathcal{C}$  the diagonal  $\delta_A : [0, 1] \rightarrow [0, 1]$  is defined by  $\delta_A(x) = A(x, x)$ . For further background on copulas we refer to [2] and [9].

To keep notation simple and consistent throughout the paper we will let  $A, B, C$  (and the corresponding versions with subindices) denote copulas,  $E, U, V, S$  will denote Borel sets,  $H$  will denote two-dimensional distribution functions,  $F$  and  $G$  univariate distribution functions.

Suppose that  $(\Omega_1, d_1)$  and  $(\Omega_2, d_2)$  are metric spaces. A *Markov kernel* from  $\Omega_1$  to  $\mathcal{B}(\Omega_2)$  is a mapping  $K : \Omega_1 \times \mathcal{B}(\Omega_2) \rightarrow [0, 1]$  such that  $x \mapsto K(x, E)$  is measurable for every fixed  $E \in \mathcal{B}(\Omega_2)$  and  $E \mapsto K(x, E)$  is a probability measure for every fixed  $x \in \Omega_1$ . Given real-valued random variables  $X, Y$  on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , a Markov kernel  $K : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$  is called a *regular conditional distribution of Y given X* if for every  $E \in \mathcal{B}(\mathbb{R})$

$$K(X(\omega), E) = \mathbb{E}(\mathbf{1}_E \circ Y | X)(\omega) \tag{1}$$

holds  $\mathbb{P}$ -a.e. It is well known that for each pair  $(X, Y)$  of real-valued random variables a regular conditional distribution  $K(\cdot, \cdot)$  of  $Y$  given  $X$  exists, that  $K(\cdot, \cdot)$  is unique  $\mathbb{P}^X$ -a.s. (i.e. unique for  $\mathbb{P}^X$ -almost all  $x \in \mathbb{R}$ ) and that  $K(\cdot, \cdot)$  only depends on  $\mathbb{P}^{X \otimes Y}$ . Hence, given  $C \in \mathcal{C}$  and  $(X, Y) \sim C$ , we will denote (a version of) the regular conditional distribution of  $Y$  given  $X$  by  $K_C(\cdot, \cdot)$ , directly view it as Markov kernel from  $[0, 1]$  to  $\mathcal{B}([0, 1])$ , and refer to  $K_C(\cdot, \cdot)$  simply as *regular conditional distribution of  $C$*  or as *Markov kernel of  $C$* . Note that for every  $C \in \mathcal{C}$ , its regular conditional distribution  $K_C(\cdot, \cdot)$ , and every Borel set  $E \in \mathcal{B}([0, 1]^2)$  we have the following *disintegration* (here  $E_x := \{y \in [0, 1] : (x, y) \in E\}$  denotes the  $x$ -section of  $E$  for every  $x \in [0, 1]$ )

$$\int_{[0,1]} K_C(x, E_x) d\lambda(x) = \mu_C(E), \quad (2)$$

so in particular

$$\int_{[0,1]} K_C(x, U) d\lambda(x) = \lambda(U) \quad (3)$$

for every  $U \in \mathcal{B}([0, 1])$ . On the other hand, every Markov kernel  $K : [0, 1] \times \mathcal{B}([0, 1]) \rightarrow [0, 1]$  fulfilling (3) induces a unique element  $\mu \in \mathcal{P}_{\mathcal{C}}$  via (2). For more details and properties of conditional expectation, regular conditional distributions, and disintegration see [5] and [6]. For examples underlining the usefulness of Markov kernels in the copula setting we refer, for instance, to [3, 12, 13] as well as to [2] and the reference therein.

$\mathcal{T}$  will denote the family of all  $\lambda$ -preserving transformations on  $[0, 1]$ ,  $\mathcal{T}_b$  the subclass of all bijections in  $\mathcal{T}$ . A copula  $C$  will be called *completely dependent (mutually completely dependent)* if there exists a transformation  $h \in \mathcal{T}$  ( $h \in \mathcal{T}_b$ ) such that  $K(x, E) = \mathbf{1}_E(h(x)) = \delta_{h(x)}(E)$  is a Markov kernel of  $C$ .  $\mathcal{C}_d$  will denote the family of all completely dependent copulas,  $A_h$  will denote the completely dependent copula induced by  $h \in \mathcal{T}$ . For alternative characterizations of (mutual) complete dependence we refer to [11] and the reference therein.

The metric  $D_1$  on  $\mathcal{C}$  is defined by

$$D_1(A, B) = \int_{[0,1]} \underbrace{\int_{[0,1]} |K_A(x, [0, y]) - K_B(x, [0, y])| d\lambda(x)}_{=: \Phi_{A,B}(y)} d\lambda(y).$$

According to [11] (also see [2, 3]) the resulting metric space  $(\mathcal{C}, D_1)$  is complete and separable, convergence with respect to  $D_1$  implies convergence with respect to  $d_\infty$  (but not vice versa) and  $\max_{A,B \in \mathcal{C}} D_1(A, B) = \frac{1}{2}$ .

### 3 Maximal $D_1$ -asymmetry of completely dependent copulas

As already mentioned in the introduction, according to [7, 10]  $\max_{A \in \mathcal{C}} d_\infty(A, A^t) = \frac{1}{3}$  holds. We are now going to prove  $\max_{A \in \mathcal{C}} D_1(A, A^t) = \frac{1}{2}$  and proceed in several steps: First we minimize the function  $\alpha \mapsto D_1(\alpha A_1 + (1 - \alpha)A_2, \Pi)$  for every pair  $A_1, A_2$  of completely dependent copulas, concentrate on completely dependent copulas  $A_1, A_2$  for which  $D_1(A_1, A_2)$  is maximal, and then characterize all  $h \in \mathcal{T}_b$  such that  $D_1(A_h, A_h^t)$  is maximal.

Fix  $A_1, A_2 \in \mathcal{C}_d$ , let  $h_1, h_2$  denote the corresponding  $\lambda$ -preserving transformations and consider  $\alpha \in [0, 1]$ . Setting  $C_\alpha = \alpha A_1 + (1 - \alpha)A_2$  with  $\alpha \in [0, 1]$  we have

$$\begin{aligned} f_{h_1, h_2}(\alpha) &:= D_1(C_\alpha, \Pi) = \int_{[0,1]} \int_{[0,1]} |K_{C_\alpha}(x, [0, y]) - K_\Pi(x, [0, y])| d\lambda(x)d\lambda(y) \\ &= \int_{[0,1]} \int_{[0,1]} |\alpha \mathbf{1}_{[0,y]}(h_1(x)) + (1 - \alpha) \mathbf{1}_{[0,y]}(h_2(x)) - y| d\lambda(x)d\lambda(y). \end{aligned}$$

A straightforward calculation (see Appendix) yields

$$\begin{aligned} f_{h_1, h_2}(\alpha) &= \frac{1}{3}(1 - 3\alpha + 3\alpha^2) + 2 \int_{[0,1]} C_{h_1, h_2}(y, y) d\lambda(y) - 2 \int_{[0,1]} y C_{h_1, h_2}(y, y) d\lambda(y) \\ &\quad - 2\alpha \int_{[0,\alpha]} C_{h_1, h_2}(y, y) d\lambda(y) + 2 \int_{[0,\alpha]} y C_{h_1, h_2}(y, y) d\lambda(y) \\ &\quad - 2(1 - \alpha) \int_{[0,1-\alpha]} C_{h_1, h_2}(y, y) d\lambda(y) + 2 \int_{[0,1-\alpha]} y C_{h_1, h_2}(y, y) d\lambda(y), \end{aligned}$$

where  $C_{h_1, h_2}$  denotes the copula defined by  $C_{h_1, h_2}(x, y) = \lambda(h_1^{-1}([0, x]) \cap h_2^{-1}([0, y]))$  for all  $x, y \in [0, 1]$  (see [8]). Obviously  $f_{h_1, h_2} \in C^1([0, 1])$  and we get

$$\begin{aligned} f'_{h_1, h_2}(\alpha) &= 2\alpha - 1 - 2 \left( \alpha C_{h_1, h_2}(\alpha, \alpha) + \int_{[0,\alpha]} C_{h_1, h_2}(y, y) d\lambda(y) \right) + 2\alpha C_{h_1, h_2}(\alpha, \alpha) \\ &\quad - 2 \left( -(1 - \alpha) C_{h_1, h_2}(1 - \alpha, 1 - \alpha) - \int_{[0,1-\alpha]} C_{h_1, h_2}(y, y) d\lambda(y) \right) \\ &\quad - 2(1 - \alpha) C_{h_1, h_2}(1 - \alpha, 1 - \alpha) \\ &= 2\alpha - 1 - 2 \int_{[0,\alpha]} C_{h_1, h_2}(y, y) d\lambda(y) + 2 \int_{[0,1-\alpha]} C_{h_1, h_2}(y, y) d\lambda(y). \end{aligned}$$

Hence  $f'_{h_1, h_2} \in C^1([0, 1])$  and

$$f''_{h_1, h_2}(\alpha) = 2 - 2C_{h_1, h_2}(\alpha, \alpha) - 2C_{h_1, h_2}(1 - \alpha, 1 - \alpha) \geq 2 - 2\alpha - 2(1 - \alpha) = 0,$$

so  $f_{h_1, h_2}$  is convex on  $[0, 1]$ . Considering  $f'_{h_1, h_2}(\frac{1}{2}) = 0$  it follows that  $f_{h_1, h_2}$  attains its global minimum at the point  $\frac{1}{2}$ , and we get

$$\min_{\alpha \in [0,1]} D_1(C_\alpha, \Pi) = \frac{1}{12} + 2 \int_{[0,1]} (1 - y) C_{h_1, h_2}(y, y) d\lambda(y) - 2 \int_{[0, \frac{1}{2}]} (1 - 2y) C_{h_1, h_2}(y, y) d\lambda(y). \quad (4)$$

**Remark 3.1.** The minimum can also be attained at a point  $\alpha \neq \frac{1}{2}$ . Consider  $t \in (0, \frac{1}{2})$  and let  $h_1, h_2 \in \mathcal{T}_b$  be defined by  $h_1(x) = x$  and  $h_2(x) = (t - x) \mathbf{1}_{[0,t]}(x) + x \mathbf{1}_{[t,1]}(x)$ . Then  $C_{h_1, h_2}(y, y) = y$  for all  $y \in [t, 1]$  and for every  $\alpha \in (t, \frac{1}{2})$  we get

$$f'_{h_1, h_2}(\alpha) = 2\alpha - 1 + 2 \int_{[\alpha, 1-\alpha]} y d\lambda(y) = 2\alpha - 1 + (1 - \alpha)^2 - \alpha^2 = 0.$$

Hence  $D_1(C_\alpha, \Pi)$  is minimal for every  $\alpha \in [t, \frac{1}{2}]$ .

**Example 3.2.** Consider  $C_\alpha = \alpha M + (1 - \alpha)W$ . Then  $h_1(x) = x$  and  $h_2(x) = 1 - x$  for all  $x \in [0, 1]$  and we get  $C_{h_1, h_2}(y, y) = \lambda([0, y] \cap [1 - y, 1]) = W(y, y)$ . So

$$\begin{aligned} \min_{\alpha \in [0, 1]} D_1(C_\alpha, \Pi) &= D_1\left(\frac{M + W}{2}, \Pi\right) \\ &= \frac{1}{12} + 2 \int_{[0, 1]} (1 - y)W(y, y)d\lambda(y) - 2 \int_{[0, \frac{1}{2}]} (1 - 2y)W(y, y)d\lambda(y) \\ &= \frac{1}{12} + 2 \int_{[\frac{1}{2}, 1]} (1 - y)(2y - 1)d\lambda(y) = \frac{1}{6}. \end{aligned}$$

The value  $\frac{1}{6}$  attained by the copula  $C_{\frac{1}{2}}$  in Example 3.2 is as close as we can possibly get to  $\Pi$ . In fact, for arbitrary  $A_1, A_2 \in \mathcal{C}_d$  with corresponding transformations  $h_1, h_2 \in \mathcal{T}$  we have

$$\begin{aligned} D_1\left(\frac{A_1 + A_2}{2}, \Pi\right) &= \frac{1}{12} + 2 \int_{[0, 1]} (1 - y)C_{h_1, h_2}(y, y)d\lambda(y) - 2 \int_{[0, \frac{1}{2}]} (1 - 2y)C_{h_1, h_2}(y, y)d\lambda(y) \\ &\geq \frac{1}{12} + 2 \int_{[0, 1]} (1 - y)C_{h_1, h_2}(y, y)d\lambda(y) - 2 \int_{[0, \frac{1}{2}]} (1 - y)C_{h_1, h_2}(y, y)d\lambda(y) \\ &= \frac{1}{12} + 2 \int_{[\frac{1}{2}, 1]} (1 - y)C_{h_1, h_2}(y, y)d\lambda(y) \\ &\geq \frac{1}{12} + 2 \int_{[\frac{1}{2}, 1]} (1 - y)(2y - 1)d\lambda(y) = \frac{1}{6}. \end{aligned} \tag{5}$$

with equality if and only if  $C_{h_1, h_2}(y, y) = 0$  on  $[0, \frac{1}{2}]$  and  $C_{h_1, h_2}(y, y) = 2y - 1$  on  $y \in [\frac{1}{2}, 1]$ . The following theorem adds some equivalent conditions to this observation.

**Theorem 3.3.** *Suppose that  $h_1, h_2 \in \mathcal{T}$ . Then the following conditions are equivalent for the corresponding the corresponding completely dependent copulas  $A_1 := A_{h_1}, A_2 := A_{h_2}$ .*

- (a)  $D_1\left(\frac{A_1 + A_2}{2}, \Pi\right) = \frac{1}{6}$ ,
- (b)  $C_{h_1, h_2}(y, y) = \delta_W(y)$ ,
- (c)  $\|h_1 - h_2\|_1 = \int_{[0, 1]} |h_1(x) - h_2(x)|d\lambda(x) = \frac{1}{2}$ ,
- (d)  $D_1(A_1, A_2) = \frac{1}{2}$  (i.e.  $A_1$  and  $A_2$  have maximum  $D_1$ -distance).

*Proof.* The equivalence of (a) and (b) has already been shown, the fact that (c) and (d) are equivalent was proved in [11]. It therefore suffices to show that (b) and (c) are equivalent, which can be done as follows: Using Proposition 15 in [11] (saying that the  $D_1$ -distance of two completely dependent copulas coincides with the  $L^1$ -distance of the corresponding  $\lambda$ -preserving transformations) we get

$$\begin{aligned} \|h_1 - h_2\|_1 &= \int_{[0, 1]} \int_{[0, 1]} (\mathbf{1}_{[0, y]}(h_1(x)) - \mathbf{1}_{[0, y]}(h_2(x)))^2 d\lambda(x)d\lambda(y) \\ &= \int_{[0, 1]} \int_{[0, 1]} \left( \mathbf{1}_{h_1^{-1}([0, y])}(x) - 2 \cdot \mathbf{1}_{h_1^{-1}([0, y]) \cap h_2^{-1}([0, y])}(x) + \mathbf{1}_{h_2^{-1}([0, y])}(x) \right) d\lambda(x)d\lambda(y) \\ &= 2 \int_{[0, 1]} yd\lambda(y) - 2 \int_{[0, 1]} C_{h_1, h_2}(y, y)d\lambda(y) \\ &= 1 - 2 \int_{[0, 1]} \delta_W(y)d\lambda(y) = \frac{1}{2}, \end{aligned} \tag{6}$$

so (b) implies (c). To complete the proof set  $\delta_{12}(y) := C_{h_1, h_2}(y, y)$  and assume that  $\delta_W(y_0) < \delta_{12}(y_0)$  holds for some  $y_0 \in (0, 1)$ . Considering continuity of  $\delta_{12}$  the fact that  $\|h_1 - h_2\|_1 < \frac{1}{2}$  holds follows immediately.  $\square$

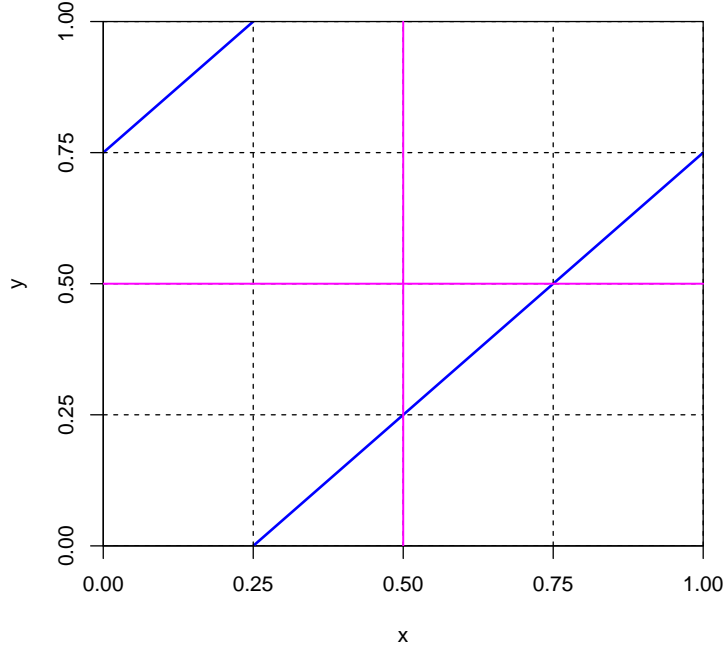


Figure 1: Completely dependent copula  $A_h$  with maximal  $D_1$ -asymmetry as considered in Example 3.4.

We now turn to the situation  $h_1 = h \in \mathcal{T}_b, h_2 = h^{-1}$  and start with the following example.

**Example 3.4.** Let  $h \in \mathcal{T}_b$  be defined by  $h(x) = x + \frac{3}{4}$  for  $x \in [0, \frac{1}{4}]$  and  $h(x) = x - \frac{1}{4}$  for  $x \in (\frac{1}{4}, 1]$  (see Figure 1). In this case we easily get  $\|h - h^{-1}\|_1 = \frac{1}{2}$ , implying  $D_1(A_h, A_h^t) = D_1(A_h, A_{h^{-1}}) = \frac{1}{2}$ , where the last equality follows from (see Lemma 10 in [11])

$$\begin{aligned} A_h^t(x, y) &= A_h(y, x) = \lambda([0, y] \cap h^{-1}([0, x])) = \lambda^{h^{-1}}([0, y] \cap h^{-1}([0, x])) \\ &= \lambda(h([0, y]) \cap [0, x]) = A_{h^{-1}}(x, y). \end{aligned}$$

Example 3.4 implies

$$\sup_{h \in \mathcal{T}_b} D_1(A_h, A_h^t) = \sup_{A \in \mathcal{C}} D_1(A, A^t) = \frac{1}{2}.$$

In other words, referring to the quantity  $\kappa(A) := 2D_1(A, A^t)$  as  $D_1$ -asymmetry of  $A \in \mathcal{C}$ , the maximum  $D_1$ -asymmetry of a copula is 1 and there exists a mutually completely dependent copula  $A_h$  with  $\kappa(A_h) = 1$ . Notice that the definition of  $\kappa : \mathcal{C} \rightarrow [0, 1]$  implies that  $\kappa(A) = 0$  if, and only if,  $A = A^t$ , i.e. if  $A$  is symmetric.

The subsequent theorem builds upon Theorem 3.3 and provides an easy characterization of all mutually completely dependent copulas  $A_h$  having maximal  $D_1$ -asymmetry.

**Theorem 3.5.** Consider  $h \in \mathcal{T}_b$ . Then the following conditions are equivalent:

- (a)  $A_h$  has maximal  $D_1$ -asymmetry (i.e.  $\kappa(A_h) = 1$ ),
- (b)  $\|h - h^{-1}\|_1 = \frac{1}{2}$ ,
- (c)  $\lambda(h^{-1}([0, \frac{1}{2}]) \cap h([0, \frac{1}{2}])) = 0$ .

*Proof.* Condition (c) implies  $C_{h,h^{-1}}(\frac{1}{2}, \frac{1}{2}) = 0$ , from which  $C_{h,h^{-1}}(y, y) = W(y, y)$  for all  $y \in [0, 1]$  follows immediately. The remaining implications are a direct consequence of Theorem 3.3.  $\square$

## 4 Maximal $D_1$ -asymmetry of general copulas

We now turn to the general situation and provide necessary and sufficient conditions for a copula  $A \in \mathcal{C}$  to have maximal  $D_1$ -asymmetry ( $\Phi_{A,A^t}$  as at the end of Section 2).

**Theorem 4.1.** For  $A \in \mathcal{C}$  the following statements are equivalent:

- (a)  $A$  has maximum  $D_1$ -asymmetry,
- (b)  $\Phi_{A,A^t}(\frac{1}{2}) = 1$ ,
- (c) There exists a Borel set  $U \in \mathcal{B}([0, 1])$  with the following properties:

$$\lambda(U \cap [0, \frac{1}{2}]) = \lambda(U \cap [\frac{1}{2}, 1]) = \frac{1}{4}, \mu_A(U \times [0, \frac{1}{2}]) = \frac{1}{2}, \mu_A([0, \frac{1}{2}] \times U) = 0. \quad (7)$$

*Proof.* The equivalence of the first and the second assertion is a direct consequence of the results in [11] (for all copulas  $A, B$  the function  $\Phi_{A,B}$  is Lipschitz-continuous with Lipschitz constant 2 and bounded from above by the tent map  $T(y) = \min\{2y, 2(1-y)\}$ ).

The fact that the second condition implies the third one can be proved as follows. Considering

$$1 = \Phi_{A,A^t}(\frac{1}{2}) = \int_{[0,1]} \underbrace{|K_A(x, [0, \frac{1}{2}]) - K_{A^t}(x, [0, \frac{1}{2}])|}_{=:g(x) \in [0,1]} d\lambda(x)$$

it follows immediately that the set  $\Lambda = \{x \in [0, 1] : g(x) = 1\}$  fulfills  $\lambda(\Lambda) = 1$ . Setting  $U := \{x \in [0, 1] : K_A(x, [0, \frac{1}{2}]) = 1\}$ , applying Scheffé's theorem ([1]) and disintegration we get

$$\begin{aligned} 1 = \Phi_{A,A^t}(\frac{1}{2}) &= \int_{[0,1]} |K_A(x, [0, \frac{1}{2}]) - K_{A^t}(x, [0, \frac{1}{2}])| d\lambda(x) \\ &= 2 \int_U (K_A(x, [0, \frac{1}{2}]) - K_{A^t}(x, [0, \frac{1}{2}])) d\lambda(x) \\ &= 2\mu_A(U \times [0, \frac{1}{2}]) - 2\mu_{A^t}(U \times [0, \frac{1}{2}]). \end{aligned} \quad (8)$$

Additionally considering

$$\begin{aligned} \mu_A(U \times [0, \frac{1}{2}]) &= \int_{U \cap \Lambda} K_A(x, [0, \frac{1}{2}]) d\lambda(x) = \int_{U \cap \Lambda} 1 d\lambda = \lambda(U \cap \Lambda) = \lambda(U), \\ \mu_A([0, \frac{1}{2}] \times U) &= \mu_{A^t}(U \times [0, \frac{1}{2}]) = \int_{U \cap \Lambda} K_{A^t}(x, [0, \frac{1}{2}]) d\lambda(x) = \int_{U \cap \Lambda} 0 d\lambda = 0 \end{aligned}$$

the last two identities in eq. (7) and  $\lambda(U) = \frac{1}{2}$  follow immediately and it remains to show that  $U$  fulfills  $\lambda(U \cap (\frac{1}{2}, 1]) = \frac{1}{4}$ . Eq. (7) implies  $\mu_A((U \times [0, \frac{1}{2}]) \setminus ([0, \frac{1}{2}] \times U)) = \frac{1}{2}$ . Hence, using  $([0, \frac{1}{2}] \times U)^c = ((\frac{1}{2}, 1] \times U) \cup ([0, 1] \times U^c)$  and the fact that  $\mu_A$  is doubly stochastic we get

$$\begin{aligned} \frac{1}{2} &= \mu_A((U \times [0, \frac{1}{2}]) \setminus ([0, \frac{1}{2}] \times U)) \\ &= \mu_A((U \cap (\frac{1}{2}, 1]) \times (U \cap [0, \frac{1}{2}])) + \mu_A(U \times ([0, \frac{1}{2}] \cap U^c)) \\ &\leq \min\{\lambda(U \cap (\frac{1}{2}, 1]), \lambda(U \cap [0, \frac{1}{2}])\} + \lambda([0, \frac{1}{2}] \cap U^c) \\ &= \min\{r, \frac{1}{2} - r\} + r =: \ell(r) \end{aligned}$$

where  $r = \lambda(U \cap (\frac{1}{2}, 1])$ . In case of  $r < \frac{1}{4}$  we would get  $\ell(r) = 2r < \frac{1}{2}$ , so  $r \geq \frac{1}{4}$  holds. Considering  $\mu_A(U \times (\frac{1}{2}, 1]) = 0$ ,  $\mu_A((\frac{1}{2}, 1] \times U) = \frac{1}{2}$  and proceeding analogously yields  $r \leq \frac{1}{4}$ , from which we finally get  $\lambda(U \cap (\frac{1}{2}, 1]) = \frac{1}{4}$ .

In case assertion three is fulfilled, disintegration implies that both  $K_A(x, [0, \frac{1}{2}]) = \mathbf{1}_U(x)$  and  $K_{A^t}(x, [0, \frac{1}{2}]) = \mathbf{1}_{U^c}(x)$  hold  $\lambda$ -a.e., from which, considering

$$\Phi_{A, A^t}(\frac{1}{2}) = \int_{[0,1]} |K_A(x, [0, \frac{1}{2}]) - K_{A^t}(x, [0, \frac{1}{2}])| d\lambda(x) = \int_{[0,1]} |\mathbf{1}_U(x) - \mathbf{1}_{U^c}(x)| d\lambda(x) = 1$$

the second assertion follows immediately.  $\square$

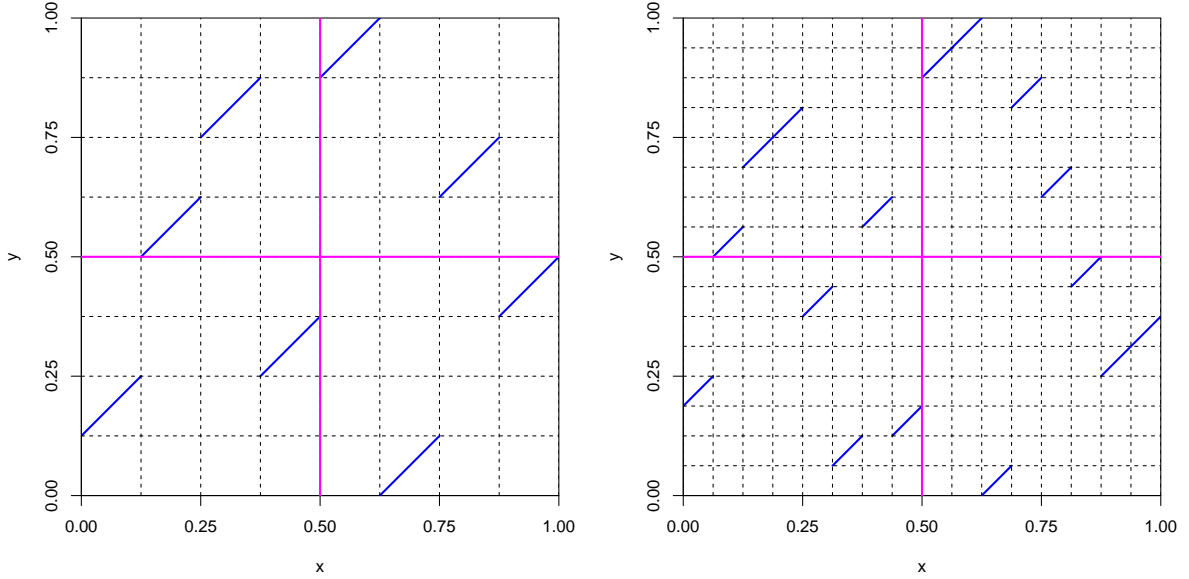


Figure 2: Shuffles with maximum  $D_1$ -asymmetry. For the shuffle in the left panel the set  $U$  is given by  $U = [0, \frac{1}{8}] \cup [\frac{3}{8}, \frac{1}{2}] \cup [\frac{5}{8}, \frac{3}{4}] \cup [\frac{7}{8}, 1]$ .

Figure 2 depicts some shuffles with maximum  $D_1$ -asymmetry. Interpreting these shuffles as checkmin copulas and replacing the minimum copula  $M$  by any other copula  $B$ , as direct consequence of Theorem 4.1 the resulting copulas have maximum  $D_1$ -asymmetry too. Assertion three in Theorem 4.1 can further be simplified, the following lemma holds:



**Lemma 4.2.** *Suppose that  $\kappa(A) = 1$  and that  $U$  is a Borel set fulfilling eq. (7). Then there exists a measurable partition of  $[0, 1]$  into sets  $U_1, V_1, U_2, V_2$  of length  $\frac{1}{4}$  fulfilling the following properties:*

1.  $U_1 \subseteq [0, \frac{1}{2}], V_1 = [0, \frac{1}{2}] \setminus U_1, U_2 \subseteq (\frac{1}{2}, 1], V_2 = (\frac{1}{2}, 1] \setminus U_2,$
2.  $\mu_A(U_1 \times V_1) = \mu_A(V_1 \times V_2) = \mu_A(U_2 \times U_1) = \mu_A(V_2 \times U_2) = \frac{1}{4}.$

Furthermore, setting  $F_i(x) := 4\lambda(U_i \cap [0, x])$  and  $G_i(x) := 4\lambda(V_i \cap [0, x])$  for  $i \in \{1, 2\}$  and  $x \in [0, 1]$ , there exist copulas  $C_1, C_2, C_3, C_4 \in \mathcal{C}$  such that the following identity holds for all  $x, y \in [0, 1]$ :

$$A(x, y) = \frac{1}{4} (C_1(F_1(x), G_1(y)) + C_2(G_1(x), G_2(y)) + C_3(F_2(x), F_1(y)) + C_4(G_2(x), F_2(y))) \quad (9)$$

*Proof.* Set  $U_1 = U \cap [0, \frac{1}{2}], U_2 = U \cap (\frac{1}{2}, 1], V = [0, 1] \setminus U, V_1 = V \cap [0, \frac{1}{2}]$  and  $V_2 = V \cap (\frac{1}{2}, 1]$ . Then eq. (7) implies  $\lambda(U_1) = \lambda(U_2) = \frac{1}{4} = \lambda(V_1) = \lambda(V_2)$  as well as

$$0 = \mu_A([0, \frac{1}{2}] \times U) = \mu_A(U_1 \times U_1) + \mu_A(U_1 \times U_2) + \mu_A(V_1 \times U_1) + \mu_A(V_1 \times U_2) \quad (10)$$

$$0 = \mu_A(V \times [0, \frac{1}{2}]) = \mu_A(V_1 \times U_1) + \mu_A(V_1 \times V_1) + \mu_A(V_2 \times U_1) + \mu_A(V_2 \times V_1). \quad (11)$$

Considering  $\frac{1}{4} = \mu_A(U_1 \times [0, \frac{1}{2}]) = \mu_A(U_1 \times U_1) + \mu_A(U_1 \times V_1)$  and using eq. (10),  $\mu_A(U_1 \times V_1) = \frac{1}{4}$  follows immediately. Additionally,  $\mu_A(V_1 \times V_2) = \frac{1}{4}$  is a direct consequence of  $\frac{1}{4} = \mu_A(V_1 \times (\frac{1}{2}, 1]) = \mu_A(V_1 \times U_2) + \mu_A(V_1 \times V_2)$  and eq. (10).

Using the fact that  $K_A(x, [0, \frac{1}{2}]) = \mathbf{1}_U(x)$  and  $K_A(x, (\frac{1}{2}, 1]) = \mathbf{1}_V(x)$  hold  $\lambda$ -a.e. and applying disintegration we get

$$\mu_A([0, \frac{1}{2}]^2) = \int_{[0, \frac{1}{2}]} K_A(x, [0, \frac{1}{2}]) d\lambda(x) = \int_{U_1} K_A(x, [0, \frac{1}{2}]) d\lambda(x) = \lambda(U_1) = \frac{1}{4} = \mu_A(U_1 \times V_1).$$

Having this,  $\mu_A((\frac{1}{2}, 1] \times [0, \frac{1}{2}]) = \frac{1}{4}$  follows and, using eq. (11), we get

$$\frac{1}{4} = \mu_A((\frac{1}{2}, 1] \times [0, \frac{1}{2}]) = \mu_A(U_2 \times U_1) + \mu_A(U_2 \times V_1).$$

Since  $\mu_A(U_2 \times V_1) > 0$  would imply

$$\frac{1}{4} = \lambda(V_1) \geq \mu_A(U \times V_1) = \mu_A(U_1 \times V_1) + \mu_A(U_2 \times V_1) > \mu_A(U_1 \times V_1) = \frac{1}{4}$$

$\mu_A(U_2 \times U_1) = \frac{1}{4}$  follows. To show  $\mu_A(V_2 \times U_2) = \frac{1}{4}$  we proceed analogously, use  $\frac{1}{4} = \mu_A(V_2 \times (\frac{1}{2}, 1])$  and the fact that  $\mu_A(V_2 \times V_2) > 0$  would imply  $\frac{1}{4} \geq \mu_A(V \times V_2) > \frac{1}{4}$ , a contradiction.

The proof of eq. (9) is now a straightforward application of Sklar's Theorem and the fact that the sets  $U_1 \times V_1, V_1 \times V_2, U_2 \times U_1, V_2 \times U_2$  are pairwise disjoint:

$$\begin{aligned} A(x, y) &= \mu_A([0, x] \times [0, y]) = \frac{1}{4} \left( 4\mu_A(U_1 \cap [0, x] \times V_1 \cap [0, y]) + 4\mu_A(V_1 \cap [0, x] \times V_2 \cap [0, y]) \right. \\ &\quad \left. + 4\mu_A(U_2 \cap [0, x] \times U_1 \cap [0, y]) + 4\mu_A(V_2 \cap [0, x] \times U_2 \cap [0, y]) \right) \\ &= \frac{1}{4} (C_1(F_1(x), G_1(y)) + C_2(G_1(x), G_2(y)) + C_3(F_2(x), F_1(y)) + C_4(G_2(x), F_2(y))) \end{aligned}$$

□

The reverse implication of Lemma 4.2 holds as well:

**Lemma 4.3.** *Suppose that  $U_1, U_2 \in \mathcal{B}([0, 1])$  fulfill  $U_1 \subseteq [0, \frac{1}{2}]$ ,  $U_2 \subseteq (\frac{1}{2}, 1]$ ,  $\lambda(U_1) = \lambda(U_2) = \frac{1}{4}$ , and let  $C_1, C_2, C_3, C_4$  be arbitrary copulas. Set  $V_1 = [0, \frac{1}{2}] \setminus U_1$ ,  $V_2 = (\frac{1}{2}, 1] \setminus U_2$  as well as  $F_i(x) := 4\lambda(U_i \cap [0, x])$  and  $G_i(x) := 4\lambda(V_i \cap [0, x])$  for  $i \in \{1, 2\}$  and  $x \in [0, 1]$ . Then the function  $A : [0, 1]^2 \rightarrow [0, 1]$ , defined by*

$$A(x, y) = \frac{1}{4} (C_1(F_1(x), G_1(y)) + C_2(G_1(x), G_2(y)) + C_3(F_2(x), F_1(y)) + C_4(G_2(x), F_2(y))), \quad (12)$$

is a copula with maximum  $D_1$ -asymmetry.

*Proof.* First of all notice that the distribution functions  $F_1, F_2, G_1, G_2$  are absolutely continuous and let  $\nu_{F_1}, \nu_{F_2}, \nu_{G_1}, \nu_{G_2}$  denote the corresponding probability measures on  $\mathcal{B}([0, 1])$ . It is clear from eq. (12) and Sklar's theorem that  $A$  is two-dimensional distribution function. Since  $A$  also fulfills the boundary conditions of a copula,  $A \in \mathcal{C}$  follows and it suffices to prove  $\kappa(A) = 1$ , which can be done as follows: Setting  $A_1(x, y) := C_1(F_1(x), G_1(y))$  yields a continuous distribution function  $A_1$ , whose corresponding probability measure  $\vartheta_{A_1}$  fulfills  $\vartheta_{A_1}([0, 1]^2) = 1$  as well as

$$\begin{aligned} \vartheta_{A_1}([\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}]) &= A_1(\bar{x}, \bar{y}) - A_1(\underline{x}, \bar{y}) - A_1(\bar{x}, \underline{y}) + A_1(\underline{x}, \underline{y}) \\ &= C_1(F_1(\bar{x}), G_1(\bar{y})) - C_1(F_1(\underline{x}), G_1(\bar{y})) - C_1(F_1(\bar{x}), G_1(\underline{y})) \\ &\quad + C_1(F_1(\underline{x}), G_1(\underline{y})) \end{aligned}$$

for all  $[\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}] \subseteq [0, 1]^2$ . Specializing to  $[\underline{y}, \bar{y}] = [0, 1]$  we get

$$\vartheta_{A_1}([\underline{x}, \bar{x}] \times [0, 1]) = C_1(F_1(\bar{x}), 1) - C_1(F_1(\underline{x}), 1) = F_1(\bar{x}) - F_1(\underline{x}) = \nu_{F_1}([\underline{x}, \bar{x}])$$

for all  $[\underline{x}, \bar{x}] \subset [0, 1]$ , implying  $\vartheta_{A_1}(U_1 \times [0, 1]) = \nu_{F_1}(U_1) = 1$ . Proceeding in the same manner yields  $\vartheta_{A_1}([0, 1] \times V_1) = \nu_{G_1}(V_1) = 1$ , from which altogether we get  $\vartheta_{A_1}(U_1 \times V_1) = 1$ . Setting  $A_2(x, y) := C_2(G_1(x), G_2(y))$ ,  $A_3(x, y) := C_3(F_2(x), F_1(y))$ ,  $A_4(x, y) := C_4(G_2(x), F_2(y))$ , and proceeding in the same manner shows  $\nu_{A_2}(V_1 \times V_2) = \nu_{A_3}(U_2 \times U_1) = \nu_{A_4}(V_2 \times U_2) = 1$ . Using the fact that  $U_1, U_2, V_1, V_2$  are pairwise disjoint yields

$$\mu_A(U_1 \times V_1) = \mu_A(V_1 \times V_2) = \mu_A(U_2 \times U_1) = \mu_A(V_2 \times U_2) = \frac{1}{4},$$

from which, setting  $U = U_1 \cup U_2$  the assertion follows as direct consequence of Theorem 4.1.  $\square$

Summing up, we have proved the following handy characterization of all copulas with maximum  $D_1$ -asymmetry.

**Theorem 4.4.** *The following statements are equivalent for  $A \in \mathcal{C}$ :*

- (a) *A has maximum  $D_1$ -asymmetry.*
- (b) *There exist sets  $U_1, U_2 \in \mathcal{B}([0, 1])$  with  $U_1 \subseteq [0, \frac{1}{2}]$ ,  $U_2 \subseteq (\frac{1}{2}, 1]$ ,  $\lambda(U_1) = \lambda(U_2) = \frac{1}{4}$ , and copulas  $C_1, C_2, C_3, C_4 \in \mathcal{C}$  such that (with the notation of Lemma 4.3) the following identity holds:*

$$\begin{aligned} A(x, y) &= \frac{1}{4} (C_1(F_1(x), G_1(y)) + C_2(G_1(x), G_2(y)) + C_3(F_2(x), F_1(y)) \\ &\quad + C_4(G_2(x), F_2(y))) \end{aligned}$$

Replacing the sets  $U_1, U_2$  in Lemma 4.2 by Borel sets  $U'_1 \subseteq [0, \frac{1}{2}], U'_2 \subseteq (\frac{1}{2}, 1]$  with  $\lambda(U_1 \Delta U'_1) = \lambda(U_2 \Delta U'_2) = 0$  obviously results in the same copula  $A$  ( $\Delta$  denoting the symmetric difference). Considering the equivalence relation  $\sim$  induced via  $(U_1, U_2) \sim (U'_1, U'_2)$  if and only if  $\lambda(U_1 \Delta U'_1) = \lambda(U_2 \Delta U'_2) = 0$  and letting  $\mathcal{E}$  denote the induced equivalence classes in

$$\{(U_1, U_2) : U_1 \in \mathcal{B}([0, \frac{1}{2}]), U_2 \in \mathcal{B}((\frac{1}{2}, 1]), \lambda(U_1) = \lambda(U_2) = \frac{1}{4}\}$$

we get the following result ( $\mathcal{C}^4 := \mathcal{C} \times \mathcal{C} \times \mathcal{C} \times \mathcal{C}$ ):

**Corollary 4.5.** *There is a one-to-one-correspondence between  $\mathcal{E} \times \mathcal{C}^4$  and  $\{A \in \mathcal{C} : \kappa(A) = 1\}$ .*

Lemma 4.3 has the following consequence:

**Corollary 4.6.** *Suppose that  $U_1 \in \mathcal{B}([0, \frac{1}{2}]), U_2 \in \mathcal{B}((\frac{1}{2}, 1])$  fulfill  $\lambda(U_1) = \lambda(U_2) = \frac{1}{4}$  and set  $V_1 = [0, \frac{1}{2}] \setminus U_1, V_2 = (\frac{1}{2}, 1] \setminus U_2$  and  $S = U_1 \times V_1 \cup V_1 \times V_2 \cup U_2 \times U_1 \cup V_2 \times U_2$ . Then there exists an absolutely continuous copula  $A$  with  $\kappa(A) = 1$  and  $\mu_A(S) = 1$ .*

*Proof.* Defining  $f : [0, 1]^2 \rightarrow [0, \infty)$  as  $f(x, y) = 4 \cdot \mathbf{1}_S(x, y)$  yields the probability density of a copula  $A$  with the desired properties.  $\square$

**Remark 4.7.** Given sets  $U_1, U_2$  with the afore-mentioned properties, Theorem 4.4 allows not only to construct absolutely continuous copulas  $A$  with  $\kappa(A) = 1$  and  $\mu_A(S) = 1$  as mentioned in Corollary 4.6. In fact, since the four copulas  $C_1, C_2, C_3, C_4$  may be chosen arbitrarily, copulas  $B$  with  $\kappa(B) = 1, \mu_B(S) = 1$  and arbitrary singular mass  $\mu_B^{sing}([0, 1]^2) \in [0, 1]$  may easily be constructed (see [13] for the definition of the singular component).

We conclude this section with an additional example illustrating that copulas with maximum  $D_1$ -asymmetry (contrary to copulas with maximal  $d_\infty$ -asymmetry) may distribute mass on the full unit square.

**Example 4.8.** There exists an absolutely continuous copula  $A$  with  $\kappa(A) = 1$  and full support  $supp(A) = [0, 1]^2$ . In fact, letting  $\Omega$  denote the set constructed in the proof of Lemma 3.1 (by starting with the Smith-Volterra-Cantor set, pasting affine copies of the set in the holes of the set and proceeding in the same manner) in [4], setting  $U_1 = \frac{1}{2}\Omega$  as well as  $U_2 = U_1 + \frac{1}{2}$ , and applying Corollary 4.6 yields a set  $S$  with  $\lambda_2(S) = \frac{1}{4}$  and  $\lambda_2(O \cap S) > 0$  for every open set  $O \subset [0, 1]^2$ . As a consequence, the corresponding (absolutely continuous) copula  $A$  fulfills  $\mu_A(O) > 0$  for every open set  $O \subset [0, 1]^2$ , implying that  $A$  has full support.

## 5 The interrelation with maximal $d_\infty$ -asymmetry

In this section we study the interrelation between maximal  $D_1$ - and maximal  $d_\infty$ -asymmetry. We start with an example of a sequence  $(A_k)_{k \in \mathbb{N}}$  of copulas with maximum  $D_1$ -asymmetry that converges to  $\Pi$  with respect to  $d_\infty$ . In other words: There are copulas with maximum  $D_1$ -asymmetry whose  $d_\infty$ -asymmetry is arbitrarily close to zero.

**Example 5.1.** For  $m \in \mathbb{N}$  let the copula  $A_m$  correspond to a uniform distribution on  $4^m$  segments as depicted in Figure 3 for the cases  $m = 1, 2, 3, 4$ . More formally, if

$x \in [\frac{2j}{2^{m+1}}, \frac{2j+1}{2^{m+1}}]$  for some  $j \in \{0, 1, \dots, 2^{m-1} - 1\}$ , then the Markov kernel  $K_{A_m}(x, E)$  of  $A_m$  is given by

$$K_{A_m}(x, E) = \frac{1}{2^{m-1}} \sum_{i=0}^{2^{m-1}-1} \mathbf{1}_E \left( x - \frac{2j}{2^{m+1}} + \frac{2i+1}{2^{m+1}} \right) \quad (13)$$

and for all other cases the formulas are analogous - the probability measure  $K_{A_m}(x, \cdot)$  is a uniform discrete distribution on  $2^{m-1}$  points. Theorem 4.4 implies that  $\kappa(A_m) = 1$  holds for every  $m \in \mathbb{N}$ . Moreover the construction of  $(A_m)_{m \in \mathbb{N}}$  implies that for every point  $(x, y) \in \{0, \frac{1}{2^k}, \frac{2}{2^k}, \dots, \frac{2^k-1}{2^k}, 1\}^2$  with  $k \in \mathbb{N}$  the sequence  $(A_m(x, y))_{m \in \mathbb{N}}$  is eventually constant. In fact, for  $(x, y) = (\frac{i}{2^k}, \frac{j}{2^k})$  we have  $A_m(x, y) = \frac{ij}{4^k}$  for every  $m \geq k + 1$ . Since the set  $\Lambda = \bigcup_{k=1}^{\infty} \Lambda_k$  with  $\Lambda_k = \{0, \frac{1}{2^k}, \frac{2}{2^k}, \dots, \frac{2^k-1}{2^k}, 1\}^2$  is dense in  $[0, 1]^2$ , the sequence  $(A_m)_{m \in \mathbb{N}}$  converges on a dense set to  $\Pi$ , from which  $\lim_{n \rightarrow \infty} d_{\infty}(A_m, \Pi) = 0$  follows.

In order to show that the  $d_{\infty}$ -asymmetry of copulas  $A$  with  $\kappa(A) = 1$  is bounded from above by  $\frac{1}{4}$  we will use the following simple lemma:

**Lemma 5.2.** *Every copula  $A \in \mathcal{C}$  with  $A(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$  fulfills  $d_{\infty}(A, A^t) \leq \frac{1}{4}$ .*

*Proof.* Considering  $|A(x, y) - A^t(x, y)| = |A(y, x) - A^t(y, x)|$  it suffices to prove the result for points  $(x, y) \in [0, 1]^2$  with  $y \leq x$ . Additionally, without loss of generality, we may assume that  $A(x, y) \geq A^t(x, y)$  holds (otherwise  $A$  and  $A^t$  change place). Moreover, Lipschitz continuity of copulas implies  $|A(x, y) - A^t(x, y)| \leq \frac{1}{4}$  for all  $(x, y) \in [0, 1]^2 \setminus [\frac{1}{4}, \frac{3}{4}]^2$ . In fact, for  $(x, y) \in [0, 1] \times [0, \frac{1}{4}]$  we have  $A(x, y), A^t(x, y) \in [0, \frac{1}{4}]$ , from which  $|A(x, y) - A^t(x, y)| \leq \frac{1}{4}$  follows immediately. For  $(x, y) \in [\frac{3}{4}, 1] \times [\frac{1}{4}, 1]$  Lipschitz continuity of copulas yields  $A(x, y), A^t(x, y) \in [y - \frac{1}{4}, y]$ , implying  $|A(x, y) - A^t(x, y)| \leq \frac{1}{4}$ . The remaining case  $(x, y) \in [0, 1] \times [\frac{3}{4}, 1]$  can be handled analogously. Additionally, considering that  $A(x, y), A^t(x, y) \in [0, \frac{1}{4}]$  holds for all  $(x, y) \in [0, \frac{1}{2}]^2$  (since  $A(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$ ) the proof is complete if  $A(x, y) - A^t(x, y) \leq \frac{1}{4}$  is proved for all  $(x, y) \in \Omega_1 \cup \Omega_2$  where  $\Omega_1 = [\frac{1}{2}, \frac{3}{4}] \times [\frac{1}{4}, \frac{1}{2}]$  and  $\Omega_2$  denotes the set of all points in the triangle with vertices  $(\frac{1}{2}, \frac{1}{2}), (\frac{3}{4}, \frac{1}{2}), (\frac{3}{4}, \frac{3}{4})$ . For  $(x, y) \in \Omega_1$  we obviously have

$$A(y, x) = y - \mu_A([0, y] \times [x, 1]) \geq y - \frac{1}{4},$$

implying

$$A(x, y) - A^t(x, y) = A(x, y) - A(y, x) \leq A(x, y) - y + \frac{1}{4} \leq \frac{1}{4}.$$

In case of  $(x, y) \in \Omega_2$  (coordinate-wise) monotonicity yields  $A(y, x) \geq A(\frac{1}{2}, x) \geq A(\frac{1}{2}, y)$ . Considering

$$A(x, x) - A(x, y) - A(\frac{1}{2}, x) + A(\frac{1}{2}, y) \geq 0,$$

using Lipschitz continuity and coordinate-wise monotonicity we finally get

$$A(x, y) - A^t(x, y) = A(x, y) - A(y, x) \leq x - \frac{1}{2} + A(\frac{1}{2}, y) - A(\frac{1}{2}, y) = x - \frac{1}{2} \leq \frac{1}{4},$$

which completes the proof.  $\square$

Having this, the proof of the following theorem is straightforward.

**Theorem 5.3.** *Every copula  $A$  having maximum  $D_1$ -asymmetry satisfies  $d_{\infty}(A, A^t) \leq \frac{1}{4}$  and the upper bound  $\frac{1}{4}$  is best possible.*

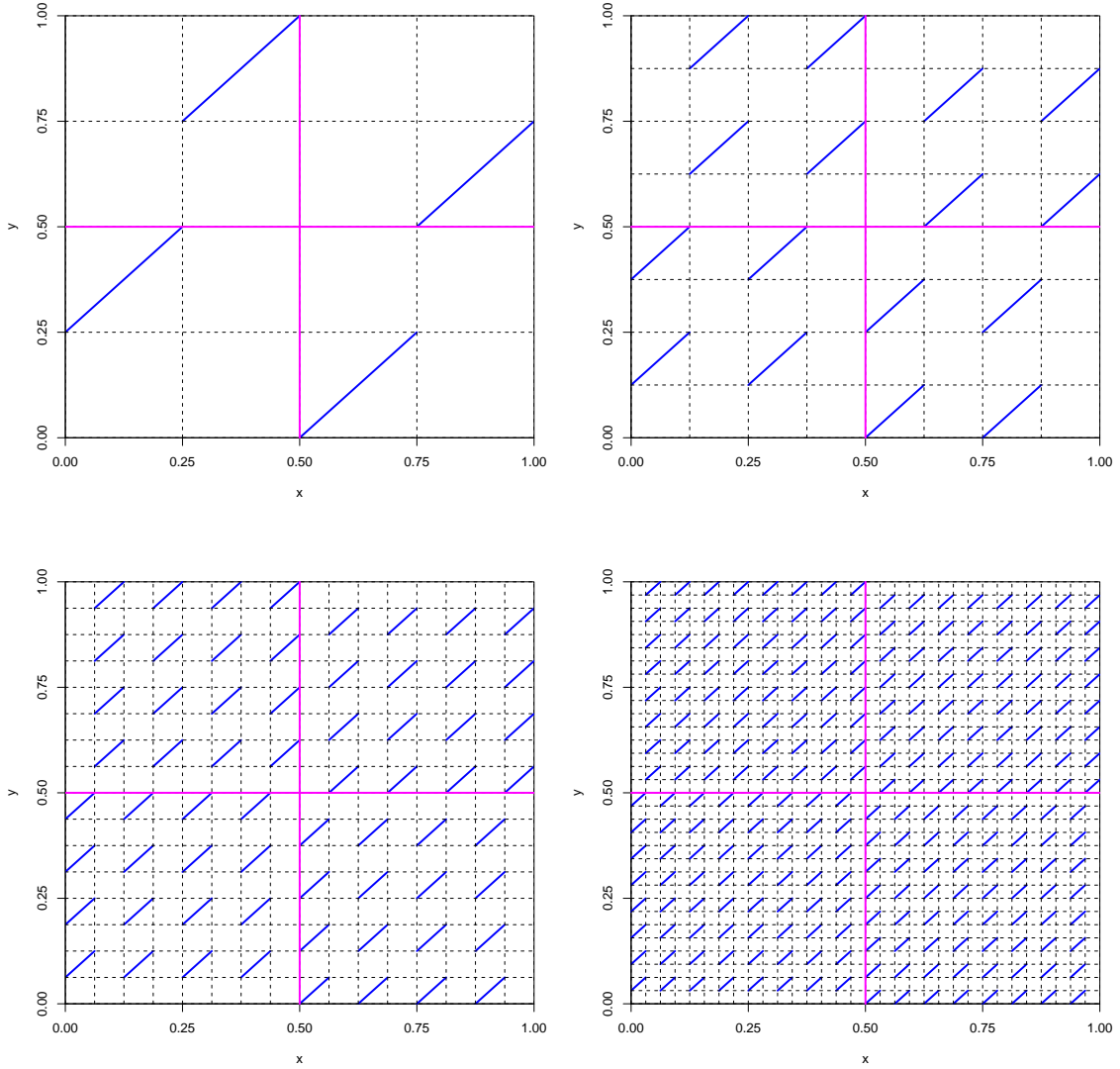


Figure 3: The copulas  $A_1, A_2, A_3, A_4$  used in Example 5.1; each of them concentrates its mass uniformly on the blue segments.

*Proof.* Suppose that  $A \in \mathcal{C}$  has maximum  $D_1$ -asymmetry. Theorem 4.4 implies  $A(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$ , so Lemma 5.2 yields  $d_\infty(A, A^t) \leq \frac{1}{4}$ . Since the completely dependent copula  $A_h$  considered in Example 3.4 obviously fulfills  $A_h(\frac{1}{2}, \frac{1}{4}) = \frac{1}{4}$  as well as  $A_h(\frac{1}{4}, \frac{1}{2}) = 0$ , the proof is complete.  $\square$

Due to the simple form of copulas with maximum  $d_\infty$ -asymmetry (see [10, 7]) it is possible to calculate the  $D_1$ -asymmetry for all these copulas.

**Theorem 5.4.** *All copulas  $A \in \mathcal{C}$  with  $d_\infty(A, A^t) = \frac{1}{3}$  fulfill  $D_1(A, A^t) = \frac{4}{9}$ .*

*Proof.* As mentioned in the introduction (also see [10])  $d_\infty(A, A^t)$  is maximal if, and only if,  $A(\frac{2}{3}, \frac{1}{3}) = \frac{1}{3}$  and  $A(\frac{1}{3}, \frac{2}{3}) = 0$  or  $A^t(\frac{2}{3}, \frac{1}{3}) = \frac{1}{3}$  and  $A^t(\frac{1}{3}, \frac{2}{3}) = 0$  holds. Without loss of generality we may consider the case  $A(\frac{1}{3}, \frac{2}{3}) = \frac{1}{3}$  and  $A(\frac{2}{3}, \frac{1}{3}) = 0$ . Using the fact that  $\mu_A$  is doubly stochastic,

$$\mu_A([0, \frac{1}{3}] \times [\frac{1}{3}, \frac{2}{3}]) = \mu_A([\frac{1}{3}, \frac{2}{3}] \times [\frac{2}{3}, 1]) = \mu_A([\frac{2}{3}, 1] \times [0, \frac{1}{3}]) = \frac{1}{3}$$

follows, and we can find copulas  $A_1, A_2, A_3 \in \mathcal{C}$  such that  $\mu_A = \frac{1}{3}\mu_{A_1}^{f_{12}} + \frac{1}{3}\mu_{A_2}^{f_{23}} + \frac{1}{3}\mu_{A_3}^{f_{31}}$  holds. Thereby the function  $f_{ij} : [0, 1]^2 \rightarrow [\frac{i-1}{3}, \frac{i}{3}] \times [\frac{j-1}{3}, \frac{j}{3}]$  is given by

$$f_{ij}(x, y) = \left(\frac{x+i-1}{3}, \frac{y+j-1}{3}\right)$$

for each  $(i, j) \in \{1, 2, 3\}^2$  and  $\mu_A^{f_{ij}}$  denotes the push-forward of  $\mu_A$  via  $f_{ij}$  for every  $A \in \mathcal{C}$ . In the same manner  $\mu_{A^t} = \frac{1}{3}\mu_{A_1^t}^{f_{21}} + \frac{1}{3}\mu_{A_2^t}^{f_{32}} + \frac{1}{3}\mu_{A_3^t}^{f_{13}}$  follows.

For  $x \in [0, \frac{1}{3}]$  we get

$$|K_A(x, [0, y]) - K_{A^t}(x, [0, y])| = \begin{cases} 0 & \text{if } y \in [0, \frac{1}{3}], \\ K_{A_1}(3x, [0, 3y-1]) & \text{if } y \in (\frac{1}{3}, \frac{2}{3}], \\ 1 - K_{A_3^t}(3x, [0, 3y-2]) & \text{if } y \in (\frac{2}{3}, 1], \end{cases}$$

for  $x \in (\frac{1}{3}, \frac{2}{3}]$

$$|K_A(x, [0, y]) - K_{A^t}(x, [0, y])| = \begin{cases} K_{A_1^t}(3x-1, [0, 3y]) & \text{if } y \in [0, \frac{1}{3}], \\ 1 & \text{if } y \in (\frac{1}{3}, \frac{2}{3}], \\ 1 - K_{A_2}(3x-1, [0, 3y-2]) & \text{if } y \in (\frac{2}{3}, 1], \end{cases}$$

follows, and for  $x \in (\frac{2}{3}, 1]$  we have

$$|K_A(x, [0, y]) - K_{A^t}(x, [0, y])| = \begin{cases} K_{A_3}(3x-2, [0, 3y]) & \text{if } y \in [0, \frac{1}{3}], \\ 1 - K_{A_2^t}(3x-2, [0, 3y-1]) & \text{if } y \in (\frac{1}{3}, \frac{2}{3}], \\ 0 & \text{if } y \in (\frac{2}{3}, 1]. \end{cases}$$

Using change of coordinates and disintegration we easily get

$$\begin{aligned} s_{1,2} &:= \int_{[\frac{1}{3}, \frac{2}{3}]} \int_{[0, \frac{1}{3}]} |K_A(x, [0, y]) - K_{A^t}(x, [0, y])| d\lambda(x) d\lambda(y) \\ &= \frac{1}{3} \int_{[\frac{1}{3}, \frac{2}{3}]} \int_{[0, 1]} K_{A_1}(x, [0, 3y-1]) d\lambda(x) d\lambda(y) = \frac{1}{3} \int_{[\frac{1}{3}, \frac{2}{3}]} (3y-1) d\lambda(y) = \frac{1}{18}. \end{aligned}$$

Proceeding analogously for

$$s_{i,j} := \int_{[\frac{j}{3}, \frac{i+1}{3}]} \int_{[\frac{i}{3}, \frac{i+1}{3}]} |K_A(x, [0, y]) - K_{A^t}(x, [0, y])| d\lambda(x) d\lambda(y)$$

with  $(i, j) \in \{1, 2, 3\}^2$  yields

$$s_{1,1} = 0, s_{1,3} = \frac{1}{18}, s_{2,1} = \frac{1}{18}, s_{2,2} = \frac{1}{9}, s_{2,3} = \frac{1}{18}, s_{3,1} = \frac{1}{18}, s_{3,2} = \frac{1}{18}, s_{3,3} = 0,$$

from which we immediately get  $D_1(A, A^t) = \frac{6}{18} + \frac{1}{9} = \frac{4}{9}$ .  $\square$

Combining Theorem 5.3 and Theorem 5.4 shows that there exists no copula having both, maximum  $d_\infty$ -asymmetry and maximum  $D_1$ -asymmetry.

**Acknowledgement:** The first author would like to thank the Development and Promotion of Science and Technology Talents Project (DPST) of the Institute for the Promotion of Teaching Science and Technology (IPST), Ministry of Education, Thailand, for the support during his study.

## Appendix

In the sequel we derive the formulas for  $f_{h_1, h_2}$  as mentioned at the beginning of Section 3. Doing so we consider four different cases and consider

$$\begin{aligned} K_{C_\alpha}(x, [0, y]) - y &= \alpha \mathbf{1}_{[0, y]}(h_1(x)) + (1 - \alpha) \mathbf{1}_{[0, y]}(h_2(x)) - y \\ &= \begin{cases} 1 - y & \text{if } x \in h_1^{-1}([0, y]) \cap h_2^{-1}([0, y]), \\ \alpha - y & \text{if } x \in h_1^{-1}([0, y]) \cap h_2^{-1}([y, 1]), \\ 1 - \alpha - y & \text{if } x \in h_1^{-1}([y, 1]) \cap h_2^{-1}([0, y]), \\ -y & \text{if } x \in h_1^{-1}([y, 1]) \cap h_2^{-1}([y, 1]). \end{cases} \end{aligned}$$

Case 1:  $x \in h_1^{-1}([0, y]) \cap h_2^{-1}([0, y])$ ,

$$\begin{aligned} &\int_{[0, 1]} \int_{[0, 1]} (1 - y) \mathbf{1}_{h_1^{-1}([0, y]) \cap h_2^{-1}([0, y])}(x) d\lambda(x) d\lambda(y) \\ &= \int_{[0, 1]} (1 - y) \lambda(h_1^{-1}([0, y]) \cap h_2^{-1}([0, y])) d\lambda(y) \\ &= \int_{[0, 1]} (1 - y) C_{h_1, h_2}(y, y) d\lambda(y) \\ &= \int_{[0, 1]} C_{h_1, h_2}(y, y) d\lambda(y) - \int_{[0, 1]} y C_{h_1, h_2}(y, y) d\lambda(y). \end{aligned}$$

Case 2:  $x \in h_1^{-1}([0, y]) \cap h_2^{-1}([y, 1])$ ,

$$\begin{aligned} &\int_{[0, 1]} \int_{[0, 1]} |\alpha - y| \mathbf{1}_{h_1^{-1}([0, y]) \cap h_2^{-1}([y, 1])}(x) d\lambda(x) d\lambda(y) \\ &= \int_{[0, 1]} |\alpha - y| \lambda(h_1^{-1}([0, y]) \cap h_2^{-1}([y, 1])) d\lambda(y) \\ &= \int_{[0, 1]} |\alpha - y| (y - C_{h_1, h_2}(y, y)) d\lambda(y) \\ &= \int_{[0, \alpha]} (\alpha - y) (y - C_{h_1, h_2}(y, y)) d\lambda(y) + \int_{(\alpha, 1]} (y - \alpha) (y - C_{h_1, h_2}(y, y)) d\lambda(y) \\ &= \int_{[0, \alpha]} (\alpha y - y^2 - \alpha C_{h_1, h_2}(y, y) + y C_{h_1, h_2}(y, y)) d\lambda(y) \\ &\quad + \int_{(\alpha, 1]} (y^2 - \alpha y + \alpha C_{h_1, h_2}(y, y) - y C_{h_1, h_2}(y, y)) d\lambda(y) \\ &= \frac{\alpha^3}{2} - \frac{\alpha^3}{3} - \alpha \int_{[0, \alpha]} C_{h_1, h_2}(y, y) d\lambda(y) + \int_{[0, \alpha]} y C_{h_1, h_2}(y, y) d\lambda(y) \\ &\quad + \frac{1 - \alpha^3}{3} - \frac{\alpha - \alpha^3}{2} + \alpha \int_{(\alpha, 1]} C_{h_1, h_2}(y, y) d\lambda(y) - \int_{(\alpha, 1]} y C_{h_1, h_2}(y, y) d\lambda(y) \\ &= \frac{1}{6} (2 - 3\alpha + 2\alpha^3) + \alpha \int_{[0, 1]} C_{h_1, h_2}(y, y) d\lambda(y) - 2\alpha \int_{[0, \alpha]} C_{h_1, h_2}(y, y) d\lambda(y) \\ &\quad - \int_{[0, 1]} y C_{h_1, h_2}(y, y) d\lambda(y) + 2 \int_{[0, \alpha]} y C_{h_1, h_2}(y, y) d\lambda(y). \end{aligned}$$

Case 3:  $x \in h_1^{-1}((y, 1]) \cap h_2^{-1}([0, y])$ ,

$$\begin{aligned}
& \int_{[0,1]} \int_{[0,1]} |1 - \alpha - y| \mathbf{1}_{h_1^{-1}((y,1]) \cap h_2^{-1}([0,y])}(x) d\lambda(x) d\lambda(y) \\
&= \int_{[0,1]} |1 - \alpha - y| \lambda(h_1^{-1}((y, 1]) \cap h_2^{-1}([0, y])) d\lambda(y) \\
&= \int_{[0,1]} |1 - \alpha - y| (y - C_{h_1, h_2}(y, y)) d\lambda(y) \\
&= \int_{[0,1-\alpha]} (1 - \alpha - y) (y - C_{h_1, h_2}(y, y)) d\lambda(y) + \int_{(1-\alpha, 1]} (y - 1 + \alpha) (y - C_{h_1, h_2}(y, y)) d\lambda(y) \\
&= \int_{[0,1-\alpha]} ((1 - \alpha)y - y^2 - (1 - \alpha)C_{h_1, h_2}(y, y) + yC_{h_1, h_2}(y, y)) d\lambda(y) \\
&\quad + \int_{(1-\alpha, 1]} (y^2 - (1 - \alpha)y + (1 - \alpha)C_{h_1, h_2}(y, y) - yC_{h_1, h_2}(y, y)) d\lambda(y) \\
&= \frac{(1 - \alpha)^3}{2} - \frac{(1 - \alpha)^3}{3} - (1 - \alpha) \int_{[0,1-\alpha]} C_{h_1, h_2}(y, y) d\lambda(y) + \int_{[0,1-\alpha]} yC_{h_1, h_2}(y, y) d\lambda(y) \\
&\quad + \frac{1 - (1 - \alpha)^3}{3} - \frac{(1 - \alpha) - (1 - \alpha)^3}{2} + (1 - \alpha) \int_{(1-\alpha, 1]} C_{h_1, h_2}(y, y) d\lambda(y) \\
&\quad - \int_{(1-\alpha, 1]} yC_{h_1, h_2}(y, y) d\lambda(y) \\
&= \frac{1}{6} (1 - 3\alpha + 6\alpha^2 - 2\alpha^3) + (1 - \alpha) \int_{[0,1]} C_{h_1, h_2}(y, y) d\lambda(y) - 2(1 - \alpha) \int_{[0,1-\alpha]} C_{h_1, h_2}(y, y) d\lambda(y) \\
&\quad - \int_{[0,1]} yC_{h_1, h_2}(y, y) d\lambda(y) + 2 \int_{[0,1-\alpha]} yC_{h_1, h_2}(y, y) d\lambda(y).
\end{aligned}$$

Case 4:  $x \in h_1^{-1}((y, 1]) \cap h_2^{-1}((y, 1])$ ,

$$\begin{aligned}
& \int_{[0,1]} \int_{[0,1]} y \mathbf{1}_{h_1^{-1}((y,1]) \cap h_2^{-1}((y,1])}(x) d\lambda(x) d\lambda(y) \\
&= \int_{[0,1]} y \lambda(h_1^{-1}((y, 1]) \cap h_2^{-1}((y, 1])) d\lambda(y) \\
&= \int_{[0,1]} y (1 - 2y + C_{h_1, h_2}(y, y)) d\lambda(y) \\
&= \int_{[0,1]} (y - 2y^2 + yC_{h_1, h_2}(y, y)) d\lambda(y) \\
&= \frac{1}{2} - \frac{2}{3} + \int_{[0,1]} yC_{h_1, h_2}(y, y) d\lambda(y) \\
&= -\frac{1}{6} + \int_{[0,1]} yC_{h_1, h_2}(y, y) d\lambda(y).
\end{aligned}$$



## References

- [1] L. Devroye: A Course in Density Estimation, Birkhäuser, Boston Basel Stuttgart, 1987.
- [2] F. Durante, C. Sempi: Principles of copula theory, CRC Press, 2015.
- [3] J. Fernández-Sánchez, W. Trutschnig: Conditioning based metrics on the space of multivariate copulas and their interrelation with uniform and levelwise convergence and Iterated Function Systems, *Journal of Theoretical Probability* 28, 1311–1336 (2015).
- [4] J. Fernández-Sánchez, W. Trutschnig: Some members of the class of (quasi-) copulas with given diagonal from the Markov kernel perspective, *Communications in Statistics Theory and Methods* 45, 1508–1526 (2016).
- [5] O. Kallenberg: Foundations of Modern Probability, Springer-Verlag, New York, Berlin, Heidelberg, 1997.
- [6] A. Klenke: Probability Theory – A Comprehensive Course, Springer-Verlag, Berlin, Heidelberg, 2007.
- [7] E.P. Klement, R. Mesiar: How non-symmetric can a copula be? *Commentationes Mathematicae Universitatis Carolinae* 47(1), 141–148 (2006).
- [8] A. Kolesárová, R. Mesiar, C. Sempi: Measure-preserving transformations, copula and compatibility, *Mediterranean Journal of Mathematics* 5, 325–339 (2008).
- [9] R.B. Nelsen: An Introduction to Copulas, Springer, New York, 2006.
- [10] R.B. Nelsen: Extremes of nonexchangeability, *Statistical Papers* 48, 329–336 (2007).
- [11] W. Trutschnig: On a strong metric on the space of copulas and its induced dependence measure, *Journal of Mathematical Analysis and Applications* 384, 690–705 (2011).
- [12] W. Trutschnig, J. Fernández-Sánchez: Idempotent and multivariate copulas with fractal support, *Journal of Statistical Planning and Inference* 142, 3086–3096 (2012).
- [13] W. Trutschnig, M. Schreyer, J. Fernández-Sánchez: Mass distributions of two-dimensional extreme-value copulas and related results, *Extremes* 19, 1–23 (2016).