HIGHLY TEMPERING INFINITE MATRICES II: FROM DIVERGENCE TO CONVERGENCE VIA TOEPLITZ–SILVERMAN MATRICES

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Abstract. It was recently proved [6] that for any Toeplitz–Silverman matrix A, there exists a dense linear subspace of the space of all sequences, all of whose nonzero elements are divergent yet whose images under A are convergent. In this paper, we improve and generalize this result by showing that, under suitable assumptions on the matrix, there are a dense set, a large algebra and a large Banach lattice consisting (except for zero) of such sequences. We show further that one of our hypotheses on the matrix A cannot in general be omitted. The case in which the field of the entries of the matrix is ultrametric is also considered.

1. Introduction

This paper extends (and provides a more precise statement) of a result from [6] concerning algebrability of certain subsets of sequences related to the class of the so-called Toeplitz-Silverman matrices. Suppose that $\mathbf{x} = (x_1, x_2, \ldots)$ is a sequence in $\mathbb{R}$ and that $A = (a_{i,j})_{i,j\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ is an infinite dimensional matrix. If $\sum_{j=1}^{\infty} a_{i,j}x_j$ exists as a real number α_i for every $i \in \mathbb{N}$ then we shall represent the sequence $(\alpha_1, \alpha_2, \ldots)$ by $A\mathbf{x}$, and we shall refer to it as the *A-transform* of $\mathbf{x}$.

Which conditions does the matrix A have to fulfill in order to assure that it preserves convergence in the sense that if $\mathbf{x}$ converges then so does $A\mathbf{x}$? Toeplitz and Silverman (see [20] or [24]) provided a full characterization of all matrices A which not only preserve convergence but also the limit, and showed that the class of such matrices coincides with the class of all matrices fulfilling the following three conditions:

(1) $\sup_{i\in\mathbb{N}} \sum_{j=1}^{\infty} |a_{i,j}| < \infty$,

(2) $\lim_{i\to\infty} \sum_{j=1}^{\infty} a_{i,j} = 1$, and

(3) $\lim_{i\to\infty} a_{i,j} = 0$ for every $j \in \mathbb{N}$.

These matrices will be referred to as TS-matrices in the sequel. We remark here that Toeplitz–Silverman's conditions also work for the complex field $\mathbb{C}$. Throughout this paper we shall only consider the real setting.

On the other hand, let us recall that a Schur matrix is an infinite matrix A with the property that, given any $\mathbf{x} \in \ell_\infty$ (the space of bounded sequences), the sequence $A\mathbf{x}$ is convergent. Let us denote by S the vector space consisting of all Schur matrices. Schur showed (see, e.g.,[3]) that if the matrix

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$A = (a_{i,j})_{i,j\in\mathbb{N}}$ satisfies that $\lim_{i\to\infty} \sum_{j=1}^{\infty} a_{i,j}$ always exists and $\sum_{j=1}^{\infty} |a_{i,j}|$ is uniformly convergent on i, then $A \in S$. This result was completed by Steinhaus, who showed that there is no matrix which is simultaneously both TS and Schur, that is, $TS \cap S = \varnothing$. Therefore, given any TS-matrix A, we can always find non-convergent bounded sequences $\mathbf{x}$ such that $A\mathbf{x}$ is also non-convergent.

Hence, the natural question is whether a TS-matrix A could also map some divergent sequences $\mathbf{x}$ to convergent ones. The authors of [6] tackled this question and showed that, given a TS-matrix A, the space of all such sequences is very large in the sense of lineability. In fact, they showed the following result:

Theorem 1.1 ([6])**.** *Assume that $A = (a_{i,j})_{i,j\geq 1}$ is an infinite matrix over the scalar field $\mathbb{R}$ satisfying the following properties:*

- (a) *For every $j \in \mathbb{N}$ the sequence $(a_{i,j})_{i\geq 1}$ converges.*
- (b) *There is a subsequence $(n_k)_{k\geq 1} \subseteq \mathbb{N}$ such that, for each subsequences $(m_k)_{k\geq 1} \subseteq (n_k)_{k\geq 1}$ we have:*
  - (b1) *the sequence $(a_{i,m_k})_{k\geq 1}$ is summable for every $i \in \mathbb{N}$, and*
  - (b2) *the sequence $\left(\sum_{k=1}^{\infty} a_{i,m_k}\right)_{i\geq 1}$ converges.*

Then there exists a dense $\mathfrak{c}$-dimensional (hence maximal dimensional) vector subspace of $\mathbb{R}^{\mathbb{N}}$ all of whose nonzero members diverge but their A-transforms converge.

As usual, we have denoted by $\mathfrak{c}$ the cardinality of the continuum, and $\mathbb{R}^{\mathbb{N}}$ stands for the topological vector space of all real sequences endowed with the product topology. If we let Ω_A denote the family of all sequences $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ for which the sequence $A\mathbf{x}$ is convergent but the sequence $\mathbf{x}$ is not, then, with the terminology of lineability (see Section 2), Theorem 1.1 tells us that, under appropriate conditions, the set Ω_A is *maximal dense-lineable.*

Our main aim in this short note is to continue on giving information about the "size" of Ω_A in terms of "minimal generating" structures. To be more precise, we shall prove a sharper version of Theorem 1.1 concerning *denseness* of Ω_A and the existence of *large algebras* as well as of *large Banach lattices* inside Ω_A, under appropriate assumptions on the matrix A. Moreover, we shall show that, if we drop the condition (a) of existence of $\lim_{i\to\infty} a_{ij}$ for every $j \in \mathbb{N}$, then, in spite of Theorem 1.1, even if condition (b) is satisfied, the denseness property does not necessarily hold.

The rest of this note is organized as follows. In Section 2 we fix some basic notation that will be used throughout the note and recall the notions of lineability, algebrability and latticeability. Additionally, we translate the conditions in Theorem 1.1 into weak convergence of signed measures. Section 3 contains the main result, its proof, and an illustrative example. Finally, in Section 4 we shall make a number of considerations regarding the ultrametric setting.

2. Preliminary notions and notations

Let us, briefly, recall the (by now) known notion of lineability and some of its variants (see, e.g., [1, 2, 4, 5, 8–12, 14, 23] for an account and state of the art of this topic). Roughly speaking, lineability is the existence of linear structures inside a family of mathematical objects that are not necessarily linear.

Let X be a vector space, α be a cardinal number, and $\Omega \subseteq X$. Then Ω is called *lineable* if there is an infinite vector space M such that $M \setminus \{0\} \subset \Omega$. If, in addition, X is a topological vector space, then Ω is called

- *α-dense-lineable* provided that there is an α-dimensional vector space M with $M \setminus \{0\} \subseteq \Omega$;
- *maximal dense-lineable* in X if it is α-dense-lineable with $\alpha = \dim(X)$.

Finally, if X if the vector space X is contained in some (linear) algebra L, them Ω is called

- *algebrable* if there is an algebra M such that $M \setminus \{0\} \subseteq \Omega$ and is infinitely generated, that is, the cardinality of any generating system is infinite;
- α-*algebrable* if there is an α-generated algebra M (that is, any minimal generating system of M has cardinality α) fulfilling $M \setminus \{0\} \subseteq \Omega$.

Of course, algebrability implies lineabiliy. The reverse implication is not true. In our terminology, algebrability equals $\aleph_0$-algebrability.

Also, let us present the notion of *latticeable* (introduced in [19] and later studied in [7, 15]). Before that, recall that a *Riesz space,* also called a vector lattice, is a partially ordered (with, say, the order $\preceq$) vector space X where the order structure is a lattice. That is, the order $\preceq$ satisfies the following properties for every pair of vectors $x, y \in X$: there is a supremum $x \vee y \in X$; for any $z \in X$ and any scalar $\alpha \geq 0$, the fact $x \preceq y$ implies $x + z \preceq y + z$ and $\alpha x \preceq \alpha y$. Then the existence of infimum $x \wedge y \in X$ is automatically satisfied; namely, $x \wedge y = -((-x) \vee (-y))$.

Definition 2.1. *Let X be a Riesz space, Ω a subset of X, and α a cardinal number. Then Ω is said to be* α-latticeable *if $\Omega \cup \{0\}$ contains a Riesz space of dimension α.*

In other order of ideas, suppose that $E \subseteq \mathbb{N}$ has cardinality $\aleph_0$ and that $\sum_{j \in E} |a_{i,j}| < \infty$ holds for every $i \in \mathbb{N}$. Then the mapping $\mu_i^E : 2^E \to \mathbb{R}$ defined by

$$\mu_i^E(J) = \sum_{j \in J} a_{i,j}$$

is a (finite) signed measure, and so we can consider its corresponding Hahn–Jordan decomposition $\mu_i^E = (\mu_i^E)^+ - (\mu_i^E)^-$ (see [13, 22]). We shall say that a sequence $(\mu_i)_{i \in \mathbb{N}}$ *converges weakly on* E (endowed with the discrete topology) to a signed measure μ on 2^E if and only if the corresponding positive and negative parts converge, it is straightforward to verify that condition (b2) in Theorem 1.1 coincides with weak convergence of $(\mu_i^E)_{i \in \mathbb{N}}$, where $E = \{n_1, n_2, n_2, \dots\}$.

3. Divergent sequences x for which $A\mathbf{x}$ is convergent

The following lemma is a slightly new version of a result from Set Theory (see [21]) and will be a crucial tool for the proof of the main result of this note. As usual, we denote $D^c := \mathbb{N} \setminus D$ if $D \subset \mathbb{N}$. Also, if D, E are sets then their symmetric difference is defined as $D \,\Delta\, E := (D \setminus E) \cup (E \setminus D)$.

Lemma 3.1. *There exists a family $\mathcal{D} = (D_t)_{t \in \mathbb{R}}$ of pairwise different subsets of $\mathbb{N}$ having the following three properties:*

(P1) *For all $n, m \in \mathbb{N}$ with $n \leq m$ and pairwise different real numbers*

$$t_1, t_2, \dots, t_n, t_{n+1}, \dots, t_m$$

we have

$$D_{t_1} \cap \ldots \cap D_{t_n} \cap D_{t_{n+1}}^c \cap \ldots \cap D_{t_m}^c \neq \varnothing. \tag{3.1}$$

(P2) *The intersections in equation* (3.1) *have cardinality* $\aleph_0$.

(P3) *For every $i \in \mathbb{N}$ we have $D_i \cap \{1, 2, \dots, i\} = \{i\}$.*

Proof. The first assertion was already established in [21]. Going through the details of the proof of the result in [21] it becomes clear to the reader that the assertion **(P2)** also holds.

In order to prove **(P3)**, we define a new family $\mathcal{D}' = (D'_t)_{t \in \mathbb{R}}$ by setting

$$D'_t = \begin{cases} D_t & t \notin \mathbb{N} \\ (D_t \setminus \{1, \dots, t-1\}) \cup \{t\} & t \in \mathbb{N}. \end{cases}$$

The construction of $\mathcal{D}'$ implies that $D_t \,\Delta\, D'_t = \varnothing$ if $t \notin \mathbb{N}$ while $D_t \,\Delta\, D'_t \subset \{1, \ldots, t\}$ if $t \in \mathbb{N}$. If $n, m \in \mathbb{N}$ with $n \le m$ and and $t_1, t_2, \ldots, t_n, t_{n+1}, \ldots, t_m$ are pairwise different real numbers then the fact that

$$B := D'_{t_1} \cap \ldots \cap D'_{t_n} \cap (D'_{t_{n+1}})^c \cap \ldots \cap (D'_{t_m})^c$$

has cardinality $\aleph_0$ is trivial for the case that $\{t_1, t_2, \ldots, t_n, t_{n+1}, \ldots, t_m\} \subseteq \mathbb{N}^c$ and straightforward for the case

$$\{t_1, t_2, \ldots, t_n, t_{n+1}, \ldots, t_m\} \cap \mathbb{N} \neq \varnothing.$$

Indeed, setting

$$\bar{t} := \max \{t_j : j \in \{1, \ldots, m\} \text{ and } t_j \in \mathbb{N}\}$$

then we have $D_{t_j} \Delta D'_{t_j} \subseteq \{1, \ldots, \bar{t}\}$ for all $j \in \{1, \ldots, m\}$. Hence, if

$$D_{t_1} \cap \ldots \cap D_{t_n} \cap D^c_{t_{n+1}} \cap \ldots \cap D^c_{t_m}$$

contains infinitely many points then so does B. This completes the proof. □

We are now ready to state and prove our main result. Observe that an infinite set $E \subseteq \mathbb{N}$ plays an important role in it. Roughly speaking, the assumptions concerning E in the next theorem tell us that matrices presenting a rather "tamed" behavior for many columns satisfy the conclusion of the theorem, even if the remaining columns behave chaotically. In the proof we shall be assuming the continuum hypothesis.

Theorem 3.2. *Let* $A = (a_{ij})_{i,j \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N} \times \mathbb{N}}$ *be a matrix such that*

- $\lim\limits_{i \to +\infty} a_{ij}$ *exists for every* $j \in \mathbb{N}$.

Suppose that there exists an infinite subset E *of* $\mathbb{N}$ *such that*

- $\sum\limits_{j \in E} |a_{i,j}| < \infty$ *for every* $i \in \mathbb{N}$.
- *The sequence* $(\mu_i^E)_{i \in \mathbb{N}}$ *of associated signed measures converges weakly on* 2^E.

Then

- Ω_A *is* $\mathfrak{c}$*-algebrable and dense in* $\mathbb{R}^{\mathbb{N}}$.
- Ω_A *is also* $\aleph_0$*-latticeable.*

Proof. Without loss of generality we may assume that E^c is countably infinite too: indeed, if this were not the case, then we would consider a subset of E having countably infinite complement and proceed analogously.

Let $\varphi : \mathbb{N} \to E$ and $\psi : \mathbb{N} \setminus \{1\} \to E^c$ denote the strictly increasing bijections between the corresponding sets, and suppose that the family $\mathcal{D} = (D_t)_{t \in \mathbb{R}}$ fulfills the three properties from Lemma 3.1. Construct a new family $\mathcal{E}^* = \{E^*_t : t \in \mathbb{R}\}$ of subsets of $\mathbb{N}$ by setting $E^*_t = \varphi(D_t)$ for every $t \in \mathbb{R}$. Notice that property **(P3)** implies that $E^*_i \cap \{1, \ldots, \varphi(i)\} = \{\varphi(i)\}$ holds for every $i \in \mathbb{N}$. Finally, define the family $\mathcal{E} = (E_t)_{t \in \mathbb{R}}$ by

$$E_t = \begin{cases} E^*_t & \text{if } t \notin \{\frac{1}{2}, \frac{1}{3}, \ldots\}, \\ \big(E^*_t \setminus \{1, 2, \ldots, \psi(\frac{1}{t}) - 1\}\big) \cup \{\psi(\frac{1}{t})\} & \text{if } t \in \{\frac{1}{2}, \frac{1}{3}, \ldots\}. \end{cases}$$

At the end of the proof we shall use the fact that the equalities

$$\min(E_i) = \varphi(i) \quad \text{and} \quad \min(E_{1/i}) = \psi(i)$$

hold for every $i \in \mathbb{N}$.

Now, for every $t \in \mathbb{R}$, let us define the binary sequence $\mathbf{s}^t = (s^t_1, s^t_2, s^t_3, \ldots)$ by

$$s^t_i = \mathbf{1}_{E_t}(i),$$

where $\mathbf{1}_C$ denotes the characteristic function of a set C. Considering that $\mathbf{s}^t$ contains infinitely many zeroes and infinitely many ones it is clear that $\mathbf{s}^t$ is not convergent.

Let us set $\mathcal{S} := \{\mathbf{s}^t : t \in \mathbb{R}\}$, a family whose cardinality is $\mathfrak{c}$. This family is linearly independent. Indeed, assume that $\lambda_1, \dots, \lambda_p, t_1, \dots, t_p$ are reals such that the t_j's are pairwise different and $\mathbf{s} := \lambda_1 \mathbf{s}^{t_1} + \cdots + \lambda_p \mathbf{s}^{t_p} = 0$. For every fixed $j \in \{1, \dots, p\}$, the set $E_{t_j} \cap \big(\bigcup_{l \in \{1,\dots,p\} \setminus \{l\}} E_{t_l}\big)^c$ has cardinality $\aleph_0$; in particular, it is nonempty, so that we can select an element l_0 in it. Then if $\mathbf{s} = (s_1, s_2, s_3, \dots)$, we get $0 = s_{l_0} = \lambda_j \cdot 1$. Hence $\lambda_j = 0$ for all j, which yields the claimed linear independence. It follows that $\dim(\operatorname{span} \mathcal{S}) = \mathfrak{c}$.

Let us now consider the algebra $\mathcal{A}(\mathcal{S})$ generated by $\mathcal{S}$. This algebra is $\mathfrak{c}$-generated because, if this were not true, then $\mathcal{A}(\mathcal{S})$ would be generated by some countable set $\mathcal{C} \subset \mathbb{R}^{\mathbb{N}}$, which means that $\mathcal{A}(\mathcal{S})$ is the linear span of the collection of all finite products of natural powers of the members of $\mathcal{C}$. But this collection is countable, which entails that the vector dimension of $\mathcal{A}(\mathcal{S})$ would be at most $\aleph_0$. This is a contradiction with the fact $\dim(\mathcal{A}(\mathcal{S})) \geq \dim(\operatorname{span} \mathcal{S}) = \mathfrak{c}$.

Thus, in order to prove that Ω_A is $\mathfrak{c}$-algebrable, our task reduces to show that every $\mathbf{s} = (s_1, s_2, s_3, \dots) \in \mathcal{A}(\mathcal{S}) \setminus \{0\}$ belongs to Ω_A. To this end, take an $\mathbf{s}$ as before. Then there exist a nonzero polynomial $p : \mathbb{R}^k \to \mathbb{R}$ without constant term and different elements $t_1, \dots, t_k \in \mathbb{R}$ such that $\mathbf{s} = p(\mathbf{s}^{t_1}, \mathbf{s}^{t_2}, \dots, \mathbf{s}^{t_k})$. Denote by F the (finite) collection of all nonempty subsets of $\{1, 2, \dots, k\}$. Then there exist constants c_J $(J \in F)$ with $c_{J_0} \neq 0$ for some $J_0 \in F$ such that $\mathbf{s}$ can be expressed as

$$\mathbf{s} = \sum_{J \in F} c_J \prod_{j \in J} \mathbf{s}^{t_j},$$

where we have used the property that every sequence $\mathbf{s}^t$ is idempotent (i.e. $(\mathbf{s}^t)^m = \mathbf{s}^t$ for all $m \in \mathbb{N}$). The construction of $\mathcal{E}$ implies that both sets

$$I_1 := \bigcap_{j \in J_0} E_{t_j} \cap \bigcap_{j \in \{1,\dots,k\} \setminus J_0} E_{t_j}^c \quad \text{and} \quad I_2 := \bigcap_{j=1}^{k} E_{t_j}^c$$

have cardinality $\aleph_0$. Moreover, for every $n \in I_1$ we have $s_n = c_{J_0} \neq 0$, while for every $n \in I_2$ we have $s_n = 0$. Hence $\mathbf{s}$ cannot be convergent.

To verify that $A\mathbf{s}$ is convergent, first notice that the i-th coordinate $(A\mathbf{s})_i$ of $A\mathbf{s}$ is given by

$$(A\mathbf{s})_i = \sum_{j=1}^{\infty} a_{i,j} s_j.$$

Let μ^E denote the signed measure to which $(\mu_i^E)_{i \in \mathbb{N}}$ converges weakly, where E is the set given in the hypothesis (see the notation in the final paragraph of Section 2). At this point, we distinguish two cases:

(i) If $\{t_1, \dots, t_k\} \subseteq \{\frac{1}{2}, \frac{1}{3}, \dots\}^c$ holds then $E_{t_i} \subseteq E$ for every $i \in \{1, \dots, k\}$ and we get

$$(A\mathbf{s})_i = \sum_{j \in E} a_{i,j} s_j = \int_E \mathbf{s}\, d\mu_i^E \overset{i \to \infty}{\longrightarrow} \int_E \mathbf{s}\, d\mu^E.$$

(ii) If $\{t_1, \dots, t_k\} \cap \{\frac{1}{2}, \frac{1}{3}, \dots\} \neq \varnothing$ then setting $m = \max\big\{\psi(\frac{1}{t_i}) : 1 \leq i \leq k,\, t_i \in \{\frac{1}{2}, \frac{1}{3}, \dots\}\big\}$ it follows that

$$(A\mathbf{s})_i = \sum_{j \in E} a_{i,j} s_j + \sum_{j \in E^c \cap \{1,\dots,m\}} a_{i,j} s_j = \underbrace{\int_E \mathbf{s}\, d\mu_i^E}_{=: x_i} + \underbrace{\sum_{j \in E^c \cap \{1,\dots,m\}} a_{i,j} s_j}_{=: y_i}.$$

The sequence $(x_i)_{i \in \mathbb{N}}$ converges to $\int_E \mathbf{s}\, d\mu^E$ and $(y_i)_{i \in \mathbb{N}}$ converges since it is a finite sum of convergent sequences. Hence $A\mathbf{s}$ converges.

Altogether we have shown that $\mathbf{s} \in \Omega_A$, as desired.

Next, in order to prove that Ω_A is dense in $\mathbb{R}^{\mathbb{N}}$, it suffices to show that the linear hull of the set

$$\left\{\mathbf{s}^t : t \in \mathbb{N} \cup \left\{\tfrac{1}{2}, \tfrac{1}{3}, \ldots\right\}\right\}$$

is dense in $\mathbb{R}^{\mathbb{N}}$. The latter property, however, is a direct consequence of the construction of the family $\mathcal{E}$. Indeed, suppose that $\mathbf{x} = (x_1, x_2, \ldots)$ is a sequence in $\mathbb{R}$. Set $\mathbf{y}^1 = x_1 \cdot \mathbf{s}^1$ if $1 \in E$ and $\mathbf{y}^1 = x_1 \cdot \mathbf{s}^{\frac{1}{2}}$ if $1 \in E^c$. By using the afore-mentioned fact that $\min(E_i) = \varphi(i)$ and $\min(E_{1/i}) = \psi(i)$ hold for every $i \in \mathbb{N}$, we can proceed analogously for all other coordinates in order to construct a sequence $(\mathbf{y}^n)_{n\in\mathbb{N}}$ which converges to $\mathbf{x}$ in the product topology.

Finally, to obtain the desired $\aleph_0$-latticeability, it suffices to divide E into subsets E_n of infinite cardinality ($n \in \mathbb{N}$). It is easy to see (we spare the reader the details) that the sequences of indicator functions of the E_n's generate a linear subspace that is, precisely, the lattice we need. □

If A is a TS-matrix then it obviously fulfills the conditions of Theorem 3.2; indeed, take $E = \mathbb{N}$, in which case the limiting measure μ^E is the zero measure. In other words, we have shown the following result:

Corollary 3.3. *If A is a TS-matrix then Ω_A is $\mathfrak{c}$-algebrable, $\aleph_0$-latticeable and dense in $\mathbb{R}^{\mathbb{N}}$.*

Let us finish this section with the following very illustrative example.

Example 3.4. *If, in Theorem 3.2, we suppressed the condition of existence of $\lim_{i\to+\infty} a_{ij}$ for every $j \in \mathbb{N}$ and we used the sequences*

$$s_i^t = \mathbf{1}_{E_t^*}(i),$$

we would still have that Ω_A is $\mathfrak{c}$-algebrable. However, it would not be possible to guarantee that the algebra we obtained is dense in $\mathbb{R}^{\mathbb{N}}$ as the following construction shows. Let A be a matrix such that

$$a_{ij} = \begin{cases} (-1)^i & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We have weak convergence of measures in $\mathbb{N} \setminus \{1\}$ since we always have the identically zero measure. We can take E to be the set of even natural integers and obtain $\mathfrak{c}$-algebrability.

Finally, let $\mathbf{s} = (s_1, \ldots, s_i, \ldots)$ be the sequence given by $s_i = (-1)^i$. We have that it is not convergent and it does not belong to the closure of Ω_A either. Indeed, if we supposed the opposite, then we would have that there is a sequence $(\mathbf{s}^n)_{n\in\mathbb{N}}$ of elements of Ω_A converging to $\mathbf{s}$. Thus, the sequence $(A\mathbf{s}^n)_{n\in\mathbb{N}} = (-1)^i s_1^n$ must converge. In other words, $s_1^n = 0$. However, this is not possible, since it must happen that $\lim_{n\to+\infty} s_1^n = -1$.

4. Final remarks on the ultrametric setting

The properties studied in the preceding sections are valid for the real field. But it may be of interest to analyze them in the case of ultrametric fields, also called nonarchimedian valued fields. In fact, infinite matrices with entries belonging to such a field have been studied in the literature (see [18]).

Recall (see e.g. [17, Chapter II]) that an *ultrametric field* is a field K equipped with a function $|\cdot| : K \to [0, +\infty)$ –called *absolute value*– with the the following properties, which hold for all $x, y \in K$:

(a) $|x| = 0$ if and only if $x = 0$.
(b) $|xy| = |x||y|$.
(c) $|x + y| \le \max\{|x|, |y|\}$.

The so-called trivial absolute value is defined as $|x| = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0. \end{cases}$

If K is an ultrametric field then the expression $d(x, y) = |x - y|$ defines a distance on K. We say that K is complete if the metric space (K, d) is complete. Note that in such a case the product space $K^{\mathbb{N}}$ is a completely metrizable topological space.

Assume that the absolute value on K is not the trivial one. As in the cases $K = \mathbb{R}$ or $\mathbb{C}$, we denote by TS the set of infinite matrices A such that, given any convergent sequence $\mathbf{x}$, the sequence $A\mathbf{x}$ exists and converges to the same limit as $\mathbf{x}$. And S will stand for the set of infinite matrices A enjoying the property that, for every bounded sequence $\mathbf{x}$, the sequence $A\mathbf{x}$ exists and converges. The basic properties of these matrices are described in detailed in [18] and we provide a brief account of them here below.

Theorem 4.1. *Assume that $A = (a_{i,j})_{i,j\geq 1}$ is an infinite matrix over the ultrametric field K with nontrivial absolute value. Then we have:*

(a) $A \in TS$ *if and only if* $\sup_{i,j\in\mathbb{N}} |a_{i,j}| < \infty$, $\lim_{i\to\infty} a_{i,j} = 0$ *for every* $j \in \mathbb{N}$ *and* $\lim_{i\to\infty} \sum_{j=1}^{\infty} a_{i,j} = 1$.

(b) $A \in S$ *if and only if* $\lim_{j\to\infty} a_{i,j} = 0$ *for every* $i \in \mathbb{N}$ *and* $\lim_{n\to\infty} \left(\sup_{j\geq 0} |a_{i+1,j} - a_{i,j}| \right) = 0$.

(c) $S \cap TS = \varnothing$.

Item (c) in the previous theorem is similar to Steinhaus' theorem for the real or complex case. Thus, the following question pops up: *Do there exist matrices A in TS satisfying that the set Ω_A is "large"?* By using the same ideas as in the previous section and with the help of this last theorem we obtain a positive answer. If we have an infinite subset $E \subset \mathbb{N}$ whose complement is also infinite and an infinite matrix A, then A_E will stand for the infinite matrix resulting after eliminating the columns whose indexes j are not in E.

We assume that K is a complete ultrametric field with nontrivial absolute value.

Theorem 4.2. *Assume that $A = (a_{i,j})_{i,j\geq 1}$ is an infinite matrix over the ultrametic field K and $E \subset \mathbb{N}$ is an infinite subset whose complement is also infinite. Let A be a matrix in $K^{\mathbb{N}\times\mathbb{N}}$. If $A_E \in S$ then we have that Ω_A is $\mathfrak{c}$-algebrable and dense in $K^{\mathbb{N}}$.*

Corollary 4.3. *Suppose that A is a TS-matrix, that $E \subseteq \mathbb{N}$ has cardinality $\aleph_0$ and that $A_E \in S$. Then Ω_A is $\mathfrak{c}$-algebrable and dense in $K^{\mathbb{N}}$.*

Recall that, following the same ideas as in the proof of Theorem 3.2, we obtain that for every ultrametric field K, the set $\ell_\infty \setminus c$ is $\mathfrak{c}$-algebrable. A similar result to the one in this very last comment, but on the real or complex setting and for the set $\ell_\infty(\Gamma) \setminus c_0(\Gamma)$ (where Γ is an infinite set), was shown in [16, Prop. 2.1].

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