On extremal problems for pairs of uniformly distributed sequences and integrals with respect to copula measures*

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Abstract

Motivated by the maximal average distance of uniformly distributed sequences we consider some extremal problems for functionals of type

$$\mu_C \mapsto \int_0^1 \int_0^1 F d\mu_C$$

where μ_C is a copula measure and F is a Riemann integrable function on $[0,1]^2$ of a specific type. Such problems have been considered in [4] and are of interest in the study of limit points of two uniformly distributed sequences.

Keywords: Uniform distribution, Copulas, Extremal problems.

AMS Subject Classification: 11K06, 62H05.

1 Introduction

According to [15], given a Riemann integrable function F defined on $[0,1]^2$ and two uniformly distributed sequences (x_n) and (y_n) in $[0,1)$, a general problem is to find limit points of the sequence (see also [10])

$$\left(\frac{1}{N} \sum_{n=1}^{N} F(x_n, y_n) \right)_{N \in \mathbb{N}}$$

As noted for instance in [4, 6] (see also [5]), this problem is equivalent to determining extreme values of the functional

$$\mu_C \mapsto \int_0^1 \int_0^1 F d\mu_C \qquad (1)$$

over all possible copula measures μ_C. Indeed, as shown in [6], this problem can be embedded in the general theory of mass transportation and optimal transport (see, e.g., [1, 11, 14]), and has important applications especially in risk management [13].

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*This version: November 13, 2020

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Here we focus on some results presented in [4] and [6] about extremes of functionals of type (1). Specifically, we present some generalizations of previous results by using different proof techniques. Although the general existence results could be also derived via optimal transport techniques [11], the present approach may provide an additional viewpoint to handle problems for uniformly distributed sequences.

2 Main results

We start by introducing some notations that will be used in the sequel.

If ϑ is a measure on the Borel σ–algebra $\mathscr{B}(\mathbb{R}^2)$ of $\mathbb{R}^2$ then G_ϑ denotes its measure-generating function defined by $G_\vartheta(x,y) = \vartheta((-\infty,x] \times (-\infty,y])$. Vice versa, for every two-dimensional measure-generating function G the corresponding measure on the Borel σ–algebra $\mathscr{B}(\mathbb{R}^2)$ will be denoted by ϑ_G or μ_G.

For every one-dimensional distribution function F the corresponding probability measure will be denoted by ξ_F and F^- denotes the pseudo- (or left-) inverse of F. Given one-dimensional distribution functions F_1, F_2 the Fréchet class of F_1, F_2, i.e. the family of all two-dimensional d.f. with marginals F_1 and F_2, will be denoted by $\mathscr{F}(F_1, F_2)$.

Finally, the class of bivariate copulas will be denoted by $\mathscr{C}$. Two elements of $\mathscr{C}$ are the independence copula Π, $\Pi(x,y) = xy$, the comonotonicity copula M, $M(x,y) = \min(x,y)$, and the countermonotonicity copula $W(x,y) = \max(x+y-1,0)$. For every $H \in \mathscr{F}(F_1,F_2)$, Sklar's Theorem ensures that there exists a copula C, which is unique if H is continuous, such that $H = C \circ (F_1, F_2)$. For more details, see [3].

The following result generalizes both Theorem 4 in [4] and Theorem 3.6 in [6] in the sense that: (a) the integral is calculated over the product $I_1 \times I_2$ of arbitrary intervals $I_1 := [a_1, b_1]$ and $I_2 := [a_2, b_2]$ of $\overline{\mathbb{R}}$ with non-empty interior, (b) the marginal distributions may be discontinuous (see Corollary 2), (c) the integrand does not need to be smooth. Additionally, the method of proof is new and is grounded on disintegration of the copula measure (see [3]). It should also be noticed that the first part of the Theorem has been essentially proved, in a different setting, in [11, Theorem 3.1.2].

Theorem 1. *Suppose that ϑ is a σ-finite (positive) measure on $I_1 \times I_2 = [a_1,b_1] \times [a_2,b_2] \subseteq \overline{\mathbb{R}}^2$ such that G_ϑ is finite on $[a_1,b_1) \times [a_2,b_2)$ and let F_1, F_2 be arbitrary continuous one-dimensional d.f.s fulfilling $F_i(b_i) - F_i(a_i) = 1$. Let $H^* \in \mathscr{F}_{F_1,F_2}$ be defined by $H^* = M \circ (F_1,F_2)$ and set $T^* := F_2^- \circ F_1$. Then we have*

$$\begin{aligned} \overline{m}_{G_\vartheta} &= \sup_{H \in \mathscr{F}(F_1,F_2)} \int_{I_1 \times I_2} G_\vartheta(x,y)\, d\mu_H(x,y) = \int_{I_1 \times I_2} G_\vartheta(x,y)\, d\mu_{H^*}(x,y) \\ &= \int_{I_1} G_\vartheta(x, T^*(x))\, d\xi_{F_1}(x). \end{aligned} \tag{2}$$

Furthermore, the following two conditions are equivalent for every continuous $H \in \mathscr{F}(F_1,F_2)$:

1. $\int_{I_1 \times I_2} G_\vartheta(x,y)\, d\mu_H(x,y) < \overline{m}_{G_\vartheta}$
2. $\vartheta\big(\{(v,w) \in I_1 \times I_2 : H(v,w) < H^*(v,w)\}\big) > 0$

Proof. Fix $H \in \mathscr{F}(F_1,F_2)$ and let $A \in \mathscr{C}$ denote the corresponding copula fulfilling $H = A \circ (F_1,F_2)$. Setting $\Gamma := \{(x,y,v,w) \in I_1 \times I_2 \times I_1 \times I_2 : v \le x, w \le y\}$ and letting $\mu_H \otimes \vartheta$ denote the product measure of μ_H and ϑ, Fubini's theorem implies

$$\mu_H \otimes \vartheta(\Gamma) = \int_{I_1 \times I_2} \vartheta(\Gamma_{(x,y)}) d\mu_H(x,y) = \int_{I_1 \times I_2} G_\vartheta(x,y) d\mu_H(x,y),$$

where $\Gamma_{(x,y)} = \{(v,w) : (x,y,v,w) \in \Gamma\}$ denotes the (x,y)-cut of Γ. Using the fact that $\Gamma_{(v,w)} = \{(x,y) \in I_1 \times I_2 : (x,y,v,w) \in \Gamma\} = [v,b_1] \times [w,b_2]$, again applying Fubini's theorem (this time in the other direction) and using continuity of F_1, F_2 and Sklar's theorem we get

$$\begin{aligned} \mu_H \otimes \vartheta(\Gamma) &= \int_{I_1 \times I_2} \mu_H(\Gamma_{(v,w)}) d\vartheta(v,w) = \int_{I_1 \times I_2} \mu_H([v,b_1] \times [w,b_2]) \, d\vartheta(v,w) \\ &= \int_{I_1 \times I_2} (1 - F_1(v) - F_2(w) + \underbrace{A(F_1(v), F_2(w))}_{\leq H^*(v,w)}) \, d\vartheta(v,w) \qquad (3) \\ &\leq \int_{I_1 \times I_2} \mu_{H^*}(\Gamma_{(v,w)}) d\vartheta(v,w) = \mu_{H^*} \otimes \vartheta(\Gamma), \end{aligned}$$

from which the first part of eq. (2) follows immediately. Considering that

$$K(x, (-\infty, y]) = K_M(F_1(x), [0, F_2(y)]) = \mathbf{1}_{[0,y]}(T^*(x))$$

is a Markov kernel of H^* (see [8]), the disintegration of the measure immediately yields

$$\int_{I_1 \times I_2} G_\vartheta(x,y) d\mu_{H^*}(x,y) = \int_{I_1} \int_{I_2} G_\vartheta(x,y) K(x,dy) d\xi_{F_1}(x) = \int_{I_1} G_\vartheta(x, T^*(x)) d\xi_{F_1}(x),$$

which completes the proof of eq. (2).
Concerning the equivalence stated in the theorem we obviously have equality in (3) if and only if

$$\int_{(a_1,b_1) \times (a_2,b_2)} A(F_1(v), F_2(w)) \, d\vartheta(v,w) = \int_{(a_1,b_1) \times (a_2,b_2)} M(F_1(v), F_2(w)) \, d\vartheta(v,w)$$

holds, from which the result follows immediately. □

Corollary 1. *Under the assumptions of Theorem 1, Eq. (2) is also valid for discontinuous F_1, F_2.*

Proof. Using eq. (3), considering

$$\begin{aligned} \mu_H([v,b_1] \times [w,b_2]) &= \lim_{n \to \infty} \mu_H\big((v - \tfrac{1}{n}, b_1] \times (w - \tfrac{1}{n}, b_2]\big) \\ &= \lim_{n \to \infty} \Big\{1 - F_1(v - \tfrac{1}{n}) - F_2(w - \tfrac{1}{n}) + \underbrace{H(v - \tfrac{1}{n}, w - \tfrac{1}{n})}_{\leq H^*\left(v - \frac{1}{n}, w - \frac{1}{n}\right)}\Big\} \\ &\leq \lim_{n \to \infty} \mu_{H^*}\big((v - \tfrac{1}{n}, b_1] \times (w - \tfrac{1}{n}, b_2]\big) = \mu_{H^*}([v,b_1] \times [w,b_2]) \end{aligned}$$

and proceeding as in the proof of Theorem 1, the desired inequality is obtained. □

In case F_1 and F_2 are assumed to be continuous, all elements $H \in \mathscr{F}(F_1, F_2)$ are continuous too. Hence, if $H(v,w) < H^*(v,w)$ holds for some $v, w \in I_1 \times I_2$, then it also holds in a neighborhood of (v,w). From this we directly get the following result.

Corollary 2. *If, under the assumption of Theorem 1, ϑ has full support, then H^* is the unique element in $\mathscr{F}(F_1, F_2)$ attaining the maximum $\overline{m}_{G_\vartheta}$.*

Example 1. Suppose that $I_1 = I_2 = [0,1]$, that ϑ is an arbitrary finite measure on $[0, \frac{3}{4}]^2$ and that $F_1(x) = F_2(x) = x\mathbf{1}_{[0,1]}(x) + \mathbf{1}_{(1,\infty)}(x)$. Furthermore let H denote the ordinal sum of M and Π with respect to the partition $\{[0, \frac{3}{4}], [\frac{3}{4}, 1]\}$ (see [9]), i.e.

$$H(x,y) = \begin{cases} \frac{3}{4} + \frac{1}{4}(4x-3)(4y-3), & (x,y) \in [3/4,1]^2, \\ \min(x,y), & \text{otherwise.} \end{cases}$$

Then obviously $\mathscr{F}(F_1, F_2) = \mathscr{C}$ and $\{(v,w) : H(v,w) \neq M(v,w)\} = (\frac{3}{4}, 1)^2$. Since $\vartheta((\frac{3}{4},1)^2) = 0$ holds, according to Theorem 1 we have

$$\overline{m}_{G_\vartheta} = \int_{[0,1]^2} G_\vartheta d\mu_H = \int_{[0,1]^2} G_\vartheta d\mu_M,$$

so the maximum $\overline{m}_{G_\vartheta}$ is attained by two different copulas, M and H.

Example 2. Assuming, as in the previous example, $I_1 = I_2 = [0,1]$, let G_ϑ be a convex combination of Π and M, that is, $G_\vartheta = \alpha\Pi + (1-\alpha)M$ for some $\alpha \in (0,1)$. First, observe that ϑ has full support, since

$$\vartheta = \alpha\mu_\Pi + (1-\alpha)\mu_M$$

and $\mu_\Pi = \lambda_2$ has full support. So, in this case, according to Theorem 1 and Corollary 2, M is the only copula attaining the maximum $\overline{m}_{G_\vartheta}$. Moreover:

$$\begin{aligned} \overline{m}_{G_\vartheta} &= \int_{[0,1]^2} \alpha\Pi + (1-\alpha)M \, d\mu_M = \alpha \int_{[0,1]} \Pi(t,t)\, d\lambda(t) + (1-\alpha) \int_{[0,1]} M(t,t)\, d\lambda(t) \\ &= \frac{\alpha}{3} + \frac{1-\alpha}{2} = \frac{1}{2} - \frac{\alpha}{6}. \end{aligned}$$

The following result provides a lower bound for integrals of type (2). It can be proved by mimicking the arguments presented in Theorem 1.

Theorem 2. *Suppose that ϑ is a σ-finite (positive) measure on $I_1 \times I_2 = [a_1, b_1] \times [a_2, b_2] \subseteq \overline{\mathbb{R}}^2$ such that G_ϑ is finite on $[a_1, b_1) \times [a_2, b_2)$ and let F_1, F_2 be arbitrary continuous one-dimensional d.f.s fulfilling $F_i(b_i) - F_i(a_i) = 1$. Let $H_* \in \mathscr{F}_{F_1,F_2}$ be defined by $H_* = W \circ (F_1, F_2)$ and set $T_* := F_2^- \circ (1 - F_1)$. Then we have*

$$\begin{aligned} \underline{m}_{G_\vartheta} &= \inf_{H \in \mathscr{F}(F_1,F_2)} \int_{I_1 \times I_2} G_\vartheta(x,y) d\mu_H(x,y) = \int_{I_1 \times I_2} G_\vartheta(x,y) d\mu_{H_*}(x,y) \\ &= \int_{I_1} G_\vartheta(x, T_*(x)) d\xi_{F_1}(x). \end{aligned} \tag{4}$$

Furthermore, the following two conditions are equivalent for every continuous $H \in \mathscr{F}(F_1, F_2)$:

1. $\int_{I_1 \times I_2} G_\vartheta(x,y) d\mu_H(x,y) > \underline{m}_{G_\vartheta}$
2. $\vartheta\big(\{(v,w) \in I_1 \times I_2 : H(v,w) > H_*(v,w)\}\big) > 0$

Now, fix $y_0 \in (0,1)$ be arbitrary and suppose both that ϑ_1 is a finite (positive) measure on $[0,1] \times [0, y_0]$ and that ϑ_2 is a finite (positive) measure on $[0,1] \times (y_0, 1]$. Setting $\vartheta = \vartheta_1 - \vartheta_2$ yields a finite signed measure on $\mathscr{B}([0,1]^2)$ with Jordan decomposition $\vartheta_1 - \vartheta_2$ (see [12]). In the following we will consider the (measurable and bounded) function $G_\vartheta(x,y) = \vartheta([0,x] \times [0,y])$. Slightly generalizing the results in [4, 6] we want to calculate

$$\overline{m}_{G_\vartheta} = \sup_{A \in \mathscr{C}} \int_{[0,1]^2} G_\vartheta \, d\mu_A. \tag{5}$$

Remark 1. Notice that, in [4, 6], instead of G_ϑ, the integrand is assumed to be a continuous function G on $[0,1]^2$ fulfilling $\frac{\partial^2 G(x,y)}{\partial y \partial x} > 0$ on $(0,1) \times (0, y_0)$ and $\frac{\partial^2 G(x,y)}{\partial y \partial x} < 0$ on $(0,1) \times (y_0, 1)$. Such a function can be considered as special cases of G_ϑ. In fact, setting

$$\begin{aligned} \vartheta_1(E \times F) &:= \int_{E \times (F \cap [0,y_0])} \frac{\partial^2 G(x,y)}{\partial y \partial x} d\lambda_2(x,y), \\ \vartheta_2(E \times F) &:= \int_{E \times (F \cap (y_0,1])} -\frac{\partial^2 G(x,y)}{\partial y \partial x} d\lambda_2(x,y), \end{aligned}$$

for $E, F \in \mathscr{B}([0,1])$ yields absolutely continuous, finite measures ϑ_1, ϑ_2. For $\vartheta = \vartheta_1 - \vartheta_2$ we obviously get $G_\vartheta = G + a$ for some constant $a \in \mathbb{R}$, implying

$$\sup_{A \in \mathscr{C}} \int_{[0,1]^2} G_\vartheta \, d\mu_A = \sup_{A \in \mathscr{C}} \int_{[0,1]^2} G \, d\mu_A + a.$$

Our general setting also includes the results from [15, Theorems 28 and 29] and [2].

As already shown in [4, Theorem 8] the problem of calculating $\overline{m}_{G_\vartheta}$ can be reduced to a one-dimensional maximization problem. Before restating the Theorem 8 in [4] and proving it in an alternative and shorter way we define the class $\mathscr{H}_{y_0}$ as the set of all y_0-sections of copulas, i.e. the family of all the maps of the form $x \mapsto C(x, y_0)$, $x \in [0,1]$, $C \in \mathscr{C}$, and state some of its properties (see [7]). First, it is clear that each element $h \in \mathscr{H}_{y_0}$ has the following properties: (i) $h(0) = 0, h(1) = y_0$, (ii) h is non-decreasing and Lipschitz continuous (with Lipschitz constant $L = 1$) and (iii) h fulfils $h(x) \in [W(x, y_0), M(x, y_0)]$. Conversely, it is straightforward to verify that each function $h : [0,1] \to [0, y_0]$ fulfilling properties (i)-(iii) is the y_0-section of a copula. In fact, the function C_h, defined by

$$C_h(x,y) = \begin{cases} M(h(x), y), & \text{if } (x,y) \in [0,1] \times [0, y_0], \\ h(x) + (1-y_0) W\left(\frac{x - h(x)}{1-y_0}, \frac{y-y_0}{1-y_0}\right), & \text{if } (x,y) \in [0,1] \times (y_0, 1]. \end{cases} \tag{6}$$

is easily shown to be a copula whose y_0-section coincides with h. Additionally, setting $\overline{h}(x) = 1 - (x - h(x))$ for every $x \in [0,1]$ as well as

$$K_{C_h}(x, E) = \mathbf{1}_E(h(x)) h'(x) + \mathbf{1}_E\left(\overline{h}(x)\right)(1 - h'(x)) \tag{7}$$

for $E \in \mathscr{B}([0,1])$ and for every $x \in [0,1]$ at which h is differentiable (recall that h is differentiable at λ-a.e. $x \in [0,1]$), it is straightforward to verify that K_{C_h} is a Markov kernel of C_h (see, e.g., [16]) and that C_h concentrates its mass on $\Gamma(h) \cup \Gamma\left(\overline{h}\right)$ (where $\Gamma(h)$ and $\Gamma\left(\overline{h}\right)$ denote the graphs of h and $\overline{h}$, respectively) in the sense that $\mu_{C_h}(\Gamma(h) \cup \Gamma\left(\overline{h}\right)) = 1$.

Theorem 3. *Consider $y_0 \in (0,1)$ and suppose that ϑ_1 is a finite (positive) measure on $[0,1]\times[0,y_0]$ and that ϑ_2 is a finite (positive) measure on $[0,1]\times(y_0,1]$. Set $\vartheta = \vartheta_1 - \vartheta_2$ and consider the (measurable and bounded) function $G_\vartheta(x,y) = \vartheta([0,x]\times[0,y])$. The following equality holds:*

$$\begin{aligned} \sup_{B\in\mathscr{C}} \int_{[0,1]^2} G_\vartheta \, d\mu_B &= \sup_{h\in\mathscr{H}_{y_0}} \int_{[0,1]^2} G_\vartheta \, d\mu_{C_h} \\ &= \sup_{h\in\mathscr{H}_{y_0}} \int_{[0,1]} G_\vartheta(x,h(x))h'(x) + G_\vartheta\left(x,\overline{h}(x)\right)(1-h'(x))\, d\lambda(x). \quad (8) \end{aligned}$$

Moreover, the former upper bound is attained (i.e., is a maximum) if and only if the latter is attained.

Proof. Let $A \in \mathscr{C}$ and set $h(x) = A(x,y_0)$ for every $x \in [0,1]$ and define two new (conditional) probability measures μ^+, μ^- by

$$\mu^+(\Omega) = \frac{1}{y_0}\mu_A\big(\Omega\cap([0,1]\times[0,y_0])\big), \quad \mu^-(\Omega) = \frac{1}{1-y_0}\mu_A\big(\Omega\cap([0,1]\times[y_0,1])\big)$$

for every $\Omega \in \mathscr{B}([0,1]^2)$. Calculating both marginals of μ^+ and applying Theorem 1 directly yields $T^* = h$ as well as

$$\int_{[0,1]\times[0,y_0]} G_\vartheta \, d\mu^+ \leq \frac{1}{y_0}\int_{[0,1]} G_\vartheta(x,h(x))h'(x)\, d\lambda(x).$$

Considering that for every $y \in (y_0,1]$ we have $G_\vartheta(x,y) = G_\vartheta(x,y_0) - \vartheta_2([0,x]\times[y_0,y])$ and setting $I = \int_{[0,1]\times[y_0,1]} G_\vartheta(x,y)\, d\mu^-(x,y)$ we get

$$\begin{aligned} I &= \int_{[0,1]\times[y_0,1]} G_\vartheta(x,y_0)\, d\mu^-(x,y) - \int_{[0,1]\times[y_0,1]} \vartheta_2([0,x]\times[y_0,y])\, d\mu^-(x,y) \\ &= \frac{1}{1-y_0}\int_{[0,1]} G_\vartheta(x,y_0)(1-h'(x))\, d\lambda(x) - \underbrace{\int_{[0,1]\times[y_0,1]} \vartheta_2([0,x]\times[y_0,y])\, d\mu^-(x,y)}_{I^*}, \end{aligned}$$

whereby the last equality follows from the disintegration of a measure and the fact that we have $K_A(x,(y_0,1]) = 1 - h'(x)$ for λ-a.e. $x \in [0,1]$. Calculating the marginals of μ^- and applying Theorem 2 to I^* we get $T_* = \overline{h}$ as well as

$$I^* \geq \frac{1}{1-y_0}\int_{[0,1]} \vartheta_2([0,x]\times[y_0,\overline{h}(x)])(1-h'(x))\, d\lambda(x),$$

so, altogether

$$\begin{aligned} I &\leq \frac{1}{1-y_0}\left(\int_{[0,1]} G_\vartheta(x,y_0)(1-h'(x))\, d\lambda(x) - \int_{[0,1]} \vartheta_2([0,x]\times[y_0,\overline{h}(x)])(1-h'(x))\, d\lambda(x)\right) \\ &= \frac{1}{1-y_0}\int_{[0,1]} G_\vartheta(x,\overline{h}(x))(1-h'(x))\, d\lambda(x). \end{aligned}$$

Considering $\mu_A = y_0\mu^+ + (1-y_0)\mu^-$ we conclude that the first quantity in eq. (8) can not be greater than the third one. Since, as direct consequence of eq. (7), we have

$$\int_{[0,1]^2} G_\vartheta \, d\mu_{C_h} = \int_{[0,1]} G_\vartheta(x,h(x))h'(x) + G_\vartheta\left(x,\overline{h}(x)\right)(1-h'(x))\, d\lambda(x)$$

and since C_h is a copula with y_0-section h, the proof of eq. (8) is complete. □

In general it seems unknown if, under the assumptions of Theorem 3, there is a unique function $h \in \mathscr{H}_{y_0}$ attaining the maximum. In special cases, uniqueness is clear - in the following we consider a slightly more general version of Example 3 in [4] and show uniqueness directly (without Euler equation).

Example 3. As before suppose that $y_0 \in (0,1)$. Let ϑ_1 and ϑ_2 denote absolutely continuous measures with constant densities $a > 0$ on the rectangle $[0,1] \times [0,y_0]$ and $b > 0$ on $[0,1] \times (y_0,1]$, respectively, and set $\vartheta := \vartheta_1 - \vartheta_2$. The corresponding function G_ϑ is given by

$$G_\vartheta(x,y) = \begin{cases} axy, & \text{if } (x,y) \in [0,1] \times [0,y_0] \\ axy_0 - bx(y-y_0), & \text{if } (x,y) \in [0,1] \times (y_0,1]. \end{cases}$$

For arbitrary $h \in \mathscr{H}_{y_0}$ applying Theorem 3 and using integration by parts we get

$$\begin{aligned}
\int_{[0,1]^2} G_\vartheta \, d\mu_{C_h} =& a \int_{[0,1]} x h(x) h'(x) \, d\lambda(x) + \int_{[0,1]} x \left[(a+b)y_0 - b + bx - bh(x)\right](1 - h'(x)) d\lambda(x) \\
=& \frac{a}{2} \left\{ x h^2(x) \Big|_0^1 - \int_{[0,1]} h^2(x) \, d\lambda(x) \right\} \\
&+ \frac{1}{2b} \left\{ x[(a+b)y_0 - b + bx - bh(x)]^2 \Big|_0^1 - \int_{[0,1]} [(a+b)y_0 - b + bx - bh(x)]^2 \, d\lambda(x) \right\} \\
=& \frac{a}{2} y_0^2 - \frac{a}{2} \int_{[0,1]} h^2(x) \, d\lambda(x) + \frac{a^2}{2b} y_0^2 - \int_{[0,1]} \frac{\left[(a+b)y_0 - b + bx - bh(x)\right]^2}{2b} \, d\lambda(x) \\
=& \left(\frac{a}{2} + \frac{a^2}{b} \right) y_0^2 \\
&- \frac{1}{2b} \int_{[0,1]} (ab + b^2) h^2(x) - 2b[(a+b)y_0 - b + bx] h(x) + [(a+b)y_0 - b + bx]^2 \, d\lambda(x).
\end{aligned}$$

For fixed x the latter integrand becomes minimal if $h(x) := y_0 + \frac{b}{a+b}(x-1)$. The function $h_1 : x \mapsto y_0 + \frac{b}{a+b}(x-1)$ is a global minimizer of the integral which, however, only lies in $\mathscr{H}_{y_0}$ for $y_0 = \frac{b}{a+b}$. It is straightforward to verify that for $y_0 \geq \frac{b}{a+b}$ the (piecewise linear) function h, defined by

$$h(x) := \begin{cases} x, & \text{if } x \in \left[0, \dfrac{(a+b)y_0 - b}{a}\right] \\ y_0 + \frac{b}{a+b}(x-1), & \text{if } x \in \left(\dfrac{(a+b)y_0 - b}{a}, 1\right] \end{cases}$$

is the best approximation of h_1 in $\mathscr{H}_{y_0}$. Figure 1 depicts a sample of the corresponding copula C_h for the case $a = b = 1$ and $y_0 = \frac{3}{4}$.

Finally, we aim at showing that every $h \in \mathscr{H}_{y_0}$ is in fact a possible maximizer in the sense that there exists some $\vartheta = \vartheta_1 - \vartheta_2$ such that $\sup_{B \in \mathscr{C}} \int_{[0,1]^2} G_\vartheta \, d\mu_B = \int_{[0,1]^2} G_\vartheta \, d\mu_{C_h}$.

Theorem 4. *For every $h \in \mathscr{H}_{y_0}$ there exists a measure ϑ on $\mathscr{B}([0,1]^2)$ such that*

$$\sup_{B \in \mathscr{C}} \int_{[0,1]^2} G_\vartheta \, d\mu_B = \int_{[0,1]^2} G_\vartheta \, d\mu_{C_h}.$$

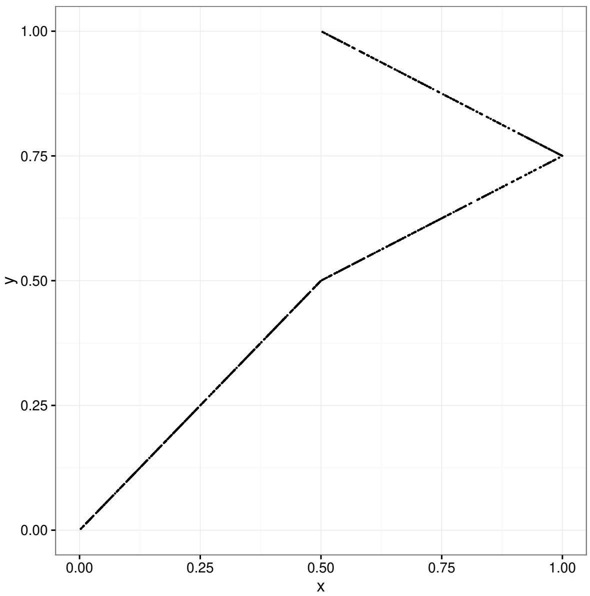

Figure 1: Sample of size $n = 1.000$ of the unique maximizer C_h in Example 3 for the case $a = b = 1$ and $y_0 = \frac{3}{4}$.

Proof. Fix $h \in \mathscr{H}_{y_0}$ and let ϑ denote the (singular) probability measure spreading its mass uniformly on $\Gamma(h)$. Thus, for every $x \in [0,1]$, we have $G_\vartheta(x, h(x)) = x$, and on the other hand $G_\vartheta\left(x, \overline{h}(x)\right) = G_\vartheta(x, y_0) = x$ as well. Hence

$$\int_{[0,1]^2} G_\vartheta \, d\mu_{C_h} = \int_{[0,1]} x \, d\lambda(x) = \frac{1}{2}.$$

Let us now consider an arbitrary $h^* \in \mathscr{H}_{y_0}$, and let $x \in [0,1]$ be any point at which both h and h^* are differentiable. If $h^*(x) < h(x)$ holds, then obviously $G_\vartheta(x, h^*(x)) < x$, whereas $G_\vartheta\left(x, \overline{h^*}(x)\right) = G_\vartheta(x, y_0) = x$, so for those values of x one has

$$G_\vartheta(x, h^*(x)) \frac{dh^*}{dx}(x) + G_\vartheta\left(x, \overline{h^*}(x)\right) \left(1 - \frac{dh^*}{dx}(x)\right) \leq x$$

where the equality holds if and only if $\dfrac{dh^*}{dx}(x) = 0$. On the other hand, if $h^*(x) > h(x)$ holds, then it is easily checked that $G_\vartheta(x, h^*(x)) = G_\vartheta\left(x, \overline{h^*}(x)\right) = x$, so for those values of x one has

$$G_\vartheta(x, h^*(x)) \frac{dh^*}{dx}(x) + G_\vartheta\left(x, \overline{h^*}(x)\right) \left(1 - \frac{dh^*}{dx}(x)\right) = x.$$

This proves that the copula C_h associated to h is a maximizer of $\int_{[0,1]^2} G_\vartheta \, d\mu_B$ over all $B \in \mathscr{C}$, and the maximum is equal to $\frac{1}{2}$. $\square$

Acknowledgements

The first author has been supported by the project “Stochastic Models for Complex Systems” by Italian MIUR (PRIN 2017, Project no. 2017JFFHSH). The fourth author gratefully acknowledges the support of the WISS 2025 project 'IDA-lab Salzburg' (20204-WISS/225/197-2019 and 0102-F1901166-KZP).

References

[1] L. Ambrosio and N. Gigli. A user's guide to optimal transport. In *Modelling and optimisation of flows on networks*, volume 2062 of *Lecture Notes in Math.*, pages 1–155. Springer, Heidelberg, 2013.

[2] V. Baláž, M. R. Iacò, O. Strauch, S. Thonhauser, and R.F. Tichy. An extremal problem in uniform distribution theory. *Unif. Distrib. Theory*, 11(2):1–21, 2016.

[3] F. Durante and C. Sempi. *Principles of copula theory*. CRC Press, Boca Raton, FL, 2016.

[4] J. Fialová and O. Strauch. On two-dimensional sequences composed by one-dimensional uniformly distributed sequences. *Unif. Distrib. Theory*, 6(1):101–125, 2011.

[5] M. Hofer and M. R. Iacò. Optimal bounds for integrals with respect to copulas and applications. *J. Optim. Theory Appl.*, 161(3):999–1011, 2014.

[6] M. R. Iacò, S. Thonhauser, and R. F. Tichy. Distribution functions, extremal limits and optimal transport. *Indag. Math. (N.S.)*, 26(5):823–841, 2015.

[7] E. P. Klement, A. Kolesárová, R. Mesiar, and C. Sempi. Copulas constructed from horizontal sections. *Comm. Statist. Theory Methods*, 36(13-16):2901–2911, 2007.

[8] T. Mroz, W. Trutschnig, and J. Fernández Sánchez. Distributions with fixed marginals maximizing the mass of the endograph of a function, 2016. arXiv:1602.05807.

[9] R. B. Nelsen. *An Introduction to Copulas*. Springer Series in Statistics. Springer, New York, second edition, 2006.

[10] F. Pillichshammer and S. Steinerberger. Average distance between consecutive points of uniformly distributed sequences. *Unif. Distrib. Theory*, 4(1):51–67, 2009.

[11] S. T. Rachev and L. Rüschendorf. *Mass transportation problems. Vol. I.* Probability and its Applications (New York). Springer-Verlag, New York, 1998. Theory.

[12] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[13] L. Rüschendorf. *Mathematical risk analysis*. Springer Series in Operations Research and Financial Engineering. Springer, Heidelberg, 2013.

[14] F. Santambrogio. *Optimal transport for applied mathematicians*, volume 87 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser/Springer, Cham, 2015. Calculus of variations, PDEs, and modeling.

[15] O. Strauch. Some applications of distribution functions of sequences. *Unif. Distrib. Theory*, 10(2):117–183, 2015.

[16] W. Trutschnig. On a strong metric on the space of copulas and its induced dependence measure. *J. Math. Anal. Appl.*, 384(2):690–705, 2011.