Some properties of double shuffles of bivariate copulas and (extreme) copulas invariant with respect to Lüroth double shuffles

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Abstract

Considering the well-known shuffling operation in x- and in y-direction yields so-called double shuffles of bivariate copulas. We study continuity properties of the double shuffle operator $\mathcal{S}_{\boldsymbol{T}}$ induced by pairs $\boldsymbol{T} = (T_1 \times T_2)$ of measure preserving transformations on $([0,1], \mathcal{B}([0,1]), \lambda)$ on the family $\mathcal{C}$ of all bivariate copulas, analyze its interrelation with the star/Markov product, and show that for each left- and for each right-invertible copula A the set of all possible double shuffles of A is dense in $\mathcal{C}$ with respect to the uniform metric d_∞. After deriving some general properties of the set $\Omega_{\boldsymbol{T}}$ of all $\mathcal{S}_{\boldsymbol{T}}$-invariant copulas we focus on the situation where T_1, T_2 are strongly mixing and show that in this case the product copula Π is an extreme point of $\Omega_{\boldsymbol{T}}$. Moreover, motivated by a recent paper by Horanská and Sarkoci (Fuzzy Sets and Systems 378, 2018) we then study double shuffles induced by pairs of so-called Lüroth maps and derive various additional properties of $\Omega_{\boldsymbol{T}}$, including the surprising fact that $\Omega_{\boldsymbol{T}}$ contains uncountably many extreme points which (interpreted as doubly stochastic measures) are pairwise mutually singular with respect to each other and which allow for an explicit construction.

Keywords: Copula, Double shuffle, strongly mixing, fixed point, Markov product, Iterated Function System

1. Introduction

The study of shuffles of bivariate copulas probably goes back to some seminal work of Vitale (see [26, 27]) as well as Kimeldorf and Sampson [13].

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Formally, shuffles were introduced by Mikusínski et al. in 1992 (see [17]), where the authors constructed new copulas from the minimum copula M by (1) cutting $[0,1]^2$ vertically into a finite number of strips and (2) shuffling (and possibly flipping) the strips. The resulting family of all shuffles of M is easily shown to be a dense subset of the family of all bivariate copulas $\mathcal{C}$ endowed with the uniform metric d_∞, which proved very handy in the context of various problems (see [8, 18]), in particular in the study of the relationship of Kendall's τ and Spearman's ρ (see [20]). The concept of shuffling was extended by Durante et. al to arbitrary copulas $A \in \mathcal{C}$ (see [6, 7]) and later by Trutschnig and Fernández Sánchez in [23]. In the general setting, letting $T\colon [0,1] \to [0,1]$ denote a general λ-preserving transformation on $[0,1]$ the T-shuffle $\mathcal{S}_T(A)$ of $A \in \mathcal{C}$ is defined in terms of the corresponding doubly stochastic measure by

$$\mu_{\mathcal{S}_T(A)}(E \times F) \coloneqq \mu_A(T^{-1}(E) \times F)$$

for all Borel-sets E and F of $[0,1]$.

In the current paper we study the natural extension to so-called double shuffles, i.e., shuffles in both directions. Letting T_1, T_2 denote λ-preserving transformations on $[0,1]$ and setting $\boldsymbol{T} \coloneqq (T_1 \times T_2)\colon [0,1]^2 \to [0,1]^2$ where $(T_1 \times T_2)(x,y) \coloneqq (T_1(x), T_2(y))$, then for every copula $A \in \mathcal{C}$ the double shuffle $\mathcal{S}_{\boldsymbol{T}}(A)$ of A is defined by

$$\mu_{\mathcal{S}_{\boldsymbol{T}}(A)}(E \times F) = \mu_A(T_1^{-1}(E) \times T_2^{-1}(F)) \tag{1}$$

for all $E, F \in \mathcal{B}([0,1])$. It is straightforward to verify that $\mathcal{S}_{\boldsymbol{T}}(A)$ maps $\mathcal{C}$ into itself. Moreover, expressing $\mathcal{S}_{\boldsymbol{T}}(A)$ in terms of the star/Markov product allows to derive several continuity properties of the mapping $(T_1, T_2, A) \mapsto \mathcal{S}_{(T_1 \times T_2)}(A)$, whereby we consider the usual L_1-distance of λ-preserving transformations as well as the metrics d_∞ and D_1 (see [21]) on $\mathcal{C}$.

Triggered by [11] we then focus on the set $\Omega_{\boldsymbol{T}}$ of all $\mathcal{S}_{\boldsymbol{T}}$-invariant copulas, show that $\Omega_{\boldsymbol{T}}$ is convex, compact in $(\mathcal{C}, d_\infty)$ but not in $(\mathcal{C}, D_1)$. Moving away from full generality and having in mind Choquet's famous representation theorem (see [19]) we then focus on extreme points of $\Omega_{\boldsymbol{T}}$ and show that Π is an extreme point of $\Omega_{\boldsymbol{T}}$ whenever T_1, T_2 are strongly mixing (see [28]). Going one step further and considering so-called Lüroth maps as transformations T_1, T_2 opens the door to the construction of uncountably many extreme points of $\Omega_{\boldsymbol{T}}$. In fact, using a construction method studied in [11] in the context of tent maps (or reflected tent maps) and working with so-called Iterated Function Systems with Probabilities makes it possible to construct those extreme points explicitly. Using another mixing property of the induced bivariate Lüroth map $\boldsymbol{T_{a,b}}$ going back to [24] additionally allows to prove the surprising fact that these extreme points are pairwise mutually singular with respect to each other.

The rest of the paper is organized as follows. Section 2 gathers some preliminaries and notations that will be used throughout the paper. Section 3 studies general continuity properties of double-shuffles as well as the aforementioned interrelation with the star/Markov product. Section 4 focuses on the set $\Omega_{\boldsymbol{T}}$ of all $\mathcal{S}_{\boldsymbol{T}}$-invariant copulas and shows that Π is an extreme point of $\Omega_{\boldsymbol{T}}$ whenever T_1, T_2 are strongly mixing. After recalling basic properties of Iterated Function Systems with Probabilities and Lüroth transformations in Section 5 we then construct uncountably many extreme points of $\Omega_{\boldsymbol{T}}$ and show that they are pairwise mutually singular with respect to each other. Various examples and graphics illustrate the construction methods and obtained results.

2. Notation and Preliminaries

For every metric space (Ω, d) the Borel σ-field will be denoted by $\mathcal{B}(\Omega)$, λ will denote the Lebesgue measure on $\mathcal{B}(\mathbb{R})$. $\mathcal{T}$ will denote the class of all measurable λ-preserving transformations on $[0,1]$, i.e.,

$$\mathcal{T} = \{T\colon [0,1] \to [0,1] \text{ and } \lambda(T^{-1}(E)) = \lambda(E) \quad \forall E \in \mathcal{B}([0,1])\},$$

and $\mathcal{T}_p$ the subclass of all bijective $T \in \mathcal{T}$. Writing μ^f for the push-forward of a measure μ under a (measurable) transformation we have that $T \in \mathcal{T}$ if and only if $\lambda^T = \lambda$. Throughout the whole paper $\mathcal{C}$ will denote the family of all two-dimensional copulas, $\mathcal{P}$ the family of all doubly-stochastic measures (for background on copulas and doubly stochastic measures we refer to [8, 18] and the references therein). M denotes the minimum copula, Π the product copula, W the lower Fréchet Hoeffding bound. Additionally, the completely dependent copula induced by a measure-preserving transformation $T \in \mathcal{T}$ will be denoted by C_T (see [21], Definition 9). For every copula $C \in \mathcal{C}$ the corresponding doubly stochastic measure will be denoted by μ_C. As usual, $d_\infty(A, B)$ denotes the uniform metric on $\mathcal{C}$, i.e.,

$$d_\infty(A, B) := \max_{(x,y)\in[0,1]^2} |A(x, y) - B(x, y)|.$$

It is well known that $(\mathcal{C}, d_\infty)$ is a compact metric space.

We will call two probability measures μ_1, μ_2 on $\mathcal{B}(\Omega)$ singular with respect to each other (and write $\mu_1 \perp \mu_2$) if there exist disjoint Borel sets $E, F \in \mathcal{B}(\Omega)$ with $\mu_1(E) = 1 = \mu_2(F)$. Considering the Lebesgue-decomposition of $\mu_A \in \mathcal{P}$ with respect to the two-dimensional Lebesgue measure λ_2 (or, equivalently, with respect to μ_Π) we will write $\mu_A = \mu_A^{\ll} + \mu_A^{\perp}$ and denote the mass of the singular component $\mu_A^{\perp}$ by $sing(A) := \mu_A^{\perp}([0,1]^2)$ and the mass of the absolutely continuous component $\mu_A^{\ll}$ by $m_A^{\ll} := \mu_A^{\ll}([0,1]^2)$.

Given λ-preserving transformations T_1, T_2, we define the *(generalized) $(T_1 \times T_2)$-double shuffle of A* implicitly via the corresponding doubly stochastic measures by

$$\mu_{\mathcal{S}_{(T_1\times T_2)}(A)}(E \times F) := \mu_A(T_1^{-1}(E) \times T_2^{-1}(F)) = \mu_A^{T_1 \times T_2}(E \times F) \qquad (2)$$

for all $E, F \in \mathcal{B}([0,1])$. In other words: $\mu_{\mathcal{S}_{(T_1\times T_2)}(A)}$ is the push-forward of μ_A via $(T_1 \times T_2)\colon [0,1]^2 \to [0,1]^2$ with $(T_1 \times T_2)(x,y) := (T_1(x), T_2(y))$. In the next section we will derive properties of double shuffles in terms of mixing properties if the transformations $T_1, T_2 \in \mathcal{T}$. A measure-preserving transformation T on a probability space $(\Omega, \mathcal{A}, \mu)$ is called strongly mixing (see [3, 28]) if for all $E, F \in \mathcal{A}$ we have

$$\lim_{n\to\infty} \mu(T^{-n}(E) \cap F) = \mu(E)\mu(F).$$

Using standard measure-theoretic arguments we easily obtain the following useful equivalence:

Lemma 2.1. *Let T be a measure preserving transformation on a probability space $(\Omega, \mathcal{A}, \mu)$. Then T is strongly mixing if, and only if for all $f \in L^1(\Omega, \mathcal{A}, \mu)$ and $E \in \mathcal{A}$ we have*

$$\int_\Omega f(x) \cdot \big(\mathbb{1}_E \circ T^n(x)\big)\, d\mu \xrightarrow{n\to\infty} \mu(E) \cdot \int_\Omega f\, d\mu. \qquad (3)$$

In what follows Markov kernels will play an important role. A mapping $K\colon \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0,1]$ is called a Markov kernel from $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ if the mapping $x \mapsto K(x, B)$ is measurable for every fixed $B \in \mathcal{B}(\mathbb{R})$ and the mapping $B \mapsto K(x, B)$ is a probability measure for every fixed $x \in \mathbb{R}$. A Markov kernel $K\colon \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0,1]$ is called regular conditional distribution of a (real-valued) random variable Y given (another random variable) X if for every $B \in \mathcal{B}(\mathbb{R})$

$$K(X(\omega), B) = \mathbb{E}(\mathbb{1}_B \circ Y | X)(\omega)$$

holds $\mathbb{P}$-a.s. It is well known that a regular conditional distribution of Y given X exists and is unique $\mathbb{P}^X$-almost sure (where $\mathbb{P}^X$ denotes the distribution of X, i.e., the push-forward of $\mathbb{P}$ via X). For every $A \in \mathcal{C}$ (a version of) the corresponding regular conditional distribution (i.e. the regular conditional distribution of Y given X in the case that $(X, Y) \sim A$) will be denoted by $K_A(\cdot, \cdot)$. Note that for every $A \in \mathcal{C}$ and Borel sets $E, F \in \mathcal{B}([0,1])$ we have

$$\int_E K_A(x, F) d\lambda(x) = \mu_A(E \times F).$$

For more details and properties of conditional expectations and regular conditional distributions we refer to [12, 14]. Expressing copulas in terms of their corresponding regular conditional distribution leads to a metric stronger than d_∞ (see [21]) and defined by

$$D_1(A, B) := \int_{[0,1]} \int_{[0,1]} |K_A(x, [0, y]) - K_B(x, [0, y])| \, d\lambda(x) d\lambda(y).$$

It can be shown that $(\mathcal{C}, D_1)$ is a complete and separable metric space and that the (minimal) topology induced by D_1 is strictly finer than the one induced by d_∞. For more properties and extensions of D_1 we refer to [8, 9, 21].

Given $A, B \in \mathcal{C}$ a new copula denoted by $A * B$ can be constructed via the so-called star/Markov product $A * B$ (see [4]):

$$(A * B)(x, y) := \int_{[0,1]} A_{,2}(x, t) B_{1,}(t, y) \, d\lambda(t). \tag{4}$$

The star product $A * B$ is always a copula, i.e., no smoothness assumptions on A, B are required. Translating to the Markov kernel setting the star product is nothing new, it corresponds to the well known composition of Markov kernels:

Lemma 2.2 (Trutschnig and Fernández Sánchez, 2012). *Suppose that $A, B \in \mathcal{C}$ and let K_A, K_B denote the Markov kernels of A and B, respectively. Then the Markov kernel $K_A \circ K_B$, defined by*

$$(K_A \circ K_B)(x, F) := \int_{[0,1]} K_B(y, F) \, K_A(x, dy) \tag{5}$$

is a regular conditional distribution of $A * B$.

In the sequel we will use the fact that a copula B is completely dependent (see [21] and the references therein for equivalent formulations) if and only if it is left-invertible w.r.t. the star-product (see [4]), i.e., if there exists a copula A such that $A * B = M$ holds.

3. Properties of double shuffles and interrelations with the star product

We start with the following simple and useful lemma that expresses the double shuffle $\mathcal{S}_{(T_1\times T_2)}(A)$ in terms of the star product and in terms of a transformation of the corresponding Markov kernel K_A. For $T_2 := id_{[0,1]}$ the lemma coincides with Lemma 2 in [23].

Lemma 3.1. *Suppose that $T_1, T_2 \in \mathcal{T}$ and that $A \in \mathcal{C}$. Then*

$$\mathcal{S}_{(T_1\times T_2)}(A) = (C_{T_1})^t * A * C_{T_2}, \tag{6}$$

where $(C_{T_1})^t$ denotes the transpose of C_{T_1}. Furthermore, for every $F \in \mathcal{B}([0,1])$ the following equation holds for λ-almost every $x \in [0,1]$:

$$K_{\mathcal{S}_{(T_1\times T_2)}(A)}(x,F) = \frac{d}{dx}\int_{T_1^{-1}([0,x])} K_A(t, T_2^{-1}(F))\, d\lambda(t). \tag{7}$$

Proof. Equation (7) follows directly from equation (2) by setting $E := [0,x]$ and using disintegration (see [12]). To prove the first assertion we use Lemma 2.2 and proceed as follows: For every $E, F \in \mathcal{B}([0,1])$ we have

$$\begin{aligned}
K_{C^t_{T_1}*(A*C_{T_2})}(x,F) &= \int_{[0,1]} K_{A*C_{T_2}}(y,F) K_{C^t_{T_1}}(x,dy)\\
&= \int_{[0,1]}\int_{[0,1]} K_{C_{T_2}}(z,F) K_A(y,dz) K_{C^t_{T_1}}(x,dy)\\
&= \int_{[0,1]}\int_{[0,1]} \mathbb{1}_{T_2^{-1}(F)}(z) K_A(y,dz) K_{C^t_{T_1}}(x,dy)\\
&= K_{C^t_{T_1}*A}(x, T_2^{-1}(F)).
\end{aligned}$$

In the same way we obtain

$$K_{A*C_{T_1}}(x,E) = K_A(x, T_1^{-1}(E)).$$

Altogether we therefore get

$$\begin{aligned}
\mu_{\mathcal{S}_{(T_1\times T_2)}(A)}(E\times F) &= \int_{T_2^{-1}(F)} K_{A^t}(x, T_1^{-1}(E))\, d\lambda(x)\\
&= \int_{T_2^{-1}(F)} K_{A^t*C_{T_1}}(x,E)\, d\lambda(x)\\
&= \mu_{A^t*C_{T_1}}(T_2^{-1}(F)\times E) = \mu_{C^t_{T_1}*A}(E\times T_2^{-1}(F))
\end{aligned}$$

$$
\begin{aligned}
&= \int_E K_{C^t_{T_1} * A}(x, T_2^{-1}(F))\, d\lambda(x) \\
&= \int_E K_{C^t_{T_1} * (A * C_{T_2})}(x, F)\, d\lambda(x) \\
&= \mu_{C^t_{T_1} * A * C_{T_2}}(E \times F)
\end{aligned}
$$

which completes the proof. □

Following [23], we can express the Markov kernel of the double-shuffle $\mathcal{S}_{(T_1 \times T_2)}(A)$ of A with $T_1 \in \mathcal{T}_p$ and $T_2 \in \mathcal{T}$ by

$$K_{\mathcal{S}_{(T_1 \times T_2)}(A)}(x, F) = K_A(T_1^{-1}(x), T_2^{-1}(F)). \tag{8}$$

The following result gathers some properties of double shuffles concerning singularity and continuity.

Theorem 3.2. *Let $S, S_1, S_2, \ldots, T, T_1, T_2, \ldots \in \mathcal{T}$ and $A, A_1, A_2, \ldots \in \mathcal{C}$. Then the following assertions hold:*

1. *$sing(\mathcal{S}_{(S \times T)}(A)) \leq sing(A)$.*

2. *If $\lim_{n\to\infty} d_\infty(A_n, A) = 0$ and if we have $\lim_{n\to\infty} \|S_n - S\|_1 = 0$ as well as $\lim_{n\to\infty} \|T_n - T\|_1 = 0$, then*

$$\lim_{n\to\infty} d_\infty(\mathcal{S}_{(S_n \times T_n)}(A_n), \mathcal{S}_{(S \times T)}(A)) = 0.$$

3. *If $\lim_{n\to\infty} \|T_n - T\|_1 = 0$ as well as $\lim_{n\to\infty} D_1(A_n, A) = 0$ then we have*

$$\lim_{n\to\infty} D_1(\mathcal{S}_{(S \times T_n)}(A_n), \mathcal{S}_{(S \times T)}(A)) = 0.$$

If, additionally, $\lim_{n\to\infty} D_1(C^t_{S_n}, C^t_S) = 0$ holds then it follows that

$$\lim_{n\to\infty} D_1(\mathcal{S}_{(S_n \times T_n)}(A_n), \mathcal{S}_{(S \times T)}(A)) = 0.$$

Proof. The first assertion is a direct consequence of Theorem 7 in [23]. To prove the second assertion we proceed as follows: According to Lemma 3.1 we can write $\mathcal{S}_{(S \times T)}(A_n) = C^t_S * A_n * C_T$. Since the star product is continuous in each argument (but not jointly) with respect to d_∞ (see [5]) we get

$$\lim_{n\to\infty} d_\infty(\mathcal{S}_{(S \times T)}(A_n), \mathcal{S}_{(S \times T)}(A)) = 0. \tag{9}$$

According to the Portmanteau Theorem (see [25], page 6) it is enough to show that

$$\lim_{n\to\infty}\left|\int_{[0,1]^2} f(x,y)d\mu_{\mathcal{S}_{(S_n\times T_n)}(A_n)}(x,y)-\int_{[0,1]^2} f(x,y)d\mu_{\mathcal{S}_{(S\times T)}(A)}(x,y)\right|=0$$

holds for every Lipschitz continuous $f\colon [0,1]^2\to\mathbb{R}$ (the class of all these f is denoted by $\mathcal{L}([0,1]^2)$). For $f\in\mathcal{L}([0,1]^2)$, considering the definition of $\mu_{\mathcal{S}_{(S\times T)}(A)}$ and applying the triangle inequality yields

$$\begin{aligned}\left|\int_{[0,1]^2} fd\mu_{A_n}^{(S_n\times T_n)}-\int_{[0,1]^2} fd\mu_{A}^{(S\times T)}\right| &\leq \left|\int_{[0,1]^2} fd\mu_{A_n}^{(S_n\times T_n)}-\int_{[0,1]^2} fd\mu_{A_n}^{(S_n\times T)}\right|\\ &+\left|\int_{[0,1]^2} fd\mu_{A_n}^{(S_n\times T)}-\int_{[0,1]^2} fd\mu_{A_n}^{(S\times T)}\right|\\ &+\left|\int_{[0,1]^2} fd\mu_{A_n}^{(S\times T)}-\int_{[0,1]^2} fd\mu_{A}^{(S\times T)}\right|\\ &=: I_n^1+I_n^2+I_n^3.\end{aligned}$$

Using change of coordinates, letting L denote the Lipschitz constant of f, and considering $E\in\mathcal{B}([0,1])$ we obtain

$$\begin{aligned}I_n^1 &\leq \int_{[0,1]^2}\left|f(S_n(x),T_n(y))-f(S_n(x),T(y))\right|d\mu_{A_n}(x,y)\\ &\leq L\cdot\int_{[0,1]^2}\left|T_n(y)-T(y)\right|d\mu_{A_n}(x,y)\\ &= L\cdot\int_{[0,1]}\left|T_n(y)-T(y)\right|d\lambda(y)=L\cdot\|T_n-T\|_1.\end{aligned}$$

Proceeding analogously for I_n^2 yields $I_n^2\leq L\cdot\|S_n-S\|_1$. Since according to eq. (9) I_n^3 converges to 0 altogether we have shown that

$$\lim_{n\to\infty} I_n^1+I_n^2+I_n^3=0,$$

which completes the proof of the second assertion.
For showing the third assertion first recall that the star product is jointly continuous with respect to the metric D_1 (see [22] or [5]) and that L_1-convergence of the sequence $(T_n)_{n\in\mathbb{N}}$ to $T\in\mathcal{T}$ is equivalent to D_1-convergence of the sequence of completely dependent copulas C_{T_n} to C_T (again see [21]). Having this the desired assertion follows from Lemma 3.1. $\square$

The subsequent example shows that the assumption

$$\lim_{n\to\infty} D_1(C^t_{S_n}, C^t_S) = 0$$

in the third assertion of Theorem 3.2 can not be omitted or replaced by $\lim_{n\to\infty}\|S_n - S\|_1 = 0$.

Example 3.3. Let $S\colon [0,1] \to [0,1]$ be defined by $S(x) := 2x(mod 1)$ and $S_n\colon [0,1] \to [0,1]$ by

$$S_n(x) = \begin{cases} x + \frac{j-1}{n} & \text{if } x \in \left[\frac{j-1}{n}, \frac{j}{n}\right) \text{ and } j \in \left\{1, \ldots, \frac{n}{2}\right\} \\ x - 1 + \frac{j}{n} & \text{if } x \in \left[\frac{j-1}{n}, \frac{j}{n}\right) \text{ and } j \in \left\{\frac{n}{2}+1, \ldots, n\right\} \\ 1 & \text{if } x = 1 \end{cases}$$

for all $n \in 2\mathbb{N}$ (see Figure 1). Obviously the transformations S_n and S are λ-preserving. It is straightforward to verify that for every $n \in 2\mathbb{N}$,

$$\|S_n - S\|_1 = \frac{1}{2n}$$

holds, which implies $\lim_{n\to\infty}\|S_n - S\|_1 = \lim_{n\to\infty} D_1(C_{S_n}, C_S) = 0$. Since each $C^t_{S_n}$ is completely dependent and the set of all completely dependent copulas is closed in the metric space $(\mathcal{C}, D_1)$ any potential D_1-limit of the sequence $(C^t_{S_n})_{n\in\mathbb{N}}$ is completely dependent too (see [21], Proposition 15). Considering that C^t_S is not completely dependent, C^t_S can not be the limit of $(C^t_{S_n})_{n\in\mathbb{N}}$. Setting $T_n = T = id_{[0,1]}$ for every $n \in \mathbb{N}$ and considering $\mathcal{S}_{(S_n\times id_{[0,1]})}(M) = C^t_{S_n} * M * M = C^t_{S_n}$ we see that the double shuffle $\mathcal{S}_{(S_n\times id_{[0,1]})}(M)$ fails to converge to $\mathcal{S}_{(S\times id_{[0,1]})}(M) = C^t_S$ with respect to D_1.

Corollary 3.4. *For every copula A the set $\{\mathcal{S}_{(S\times T)}(A)\colon S, T \in \mathcal{T}\}$ only contains copulas whose singular component has mass at most $sing(A)$.*

Corollary 3.5. *For every left- and for every right-invertible copula $C \in \mathcal{C}$ the set*

$$\{\mathcal{S}_{(S\times T)}(C)\colon S, T \in \mathcal{T}\}$$

is dense in $(\mathcal{C}, d_\infty)$.

Proof. Suppose that the copula C is left-invertible. Then (see [21]), C can be expressed as $C = C_h$ for some $h \in \mathcal{T}$. According to [4] we can express the Minimum copula M as $M = C^t_h * C_h * C_{id}$. Since the composition of λ-preserving transformations is λ-preserving and since obviously

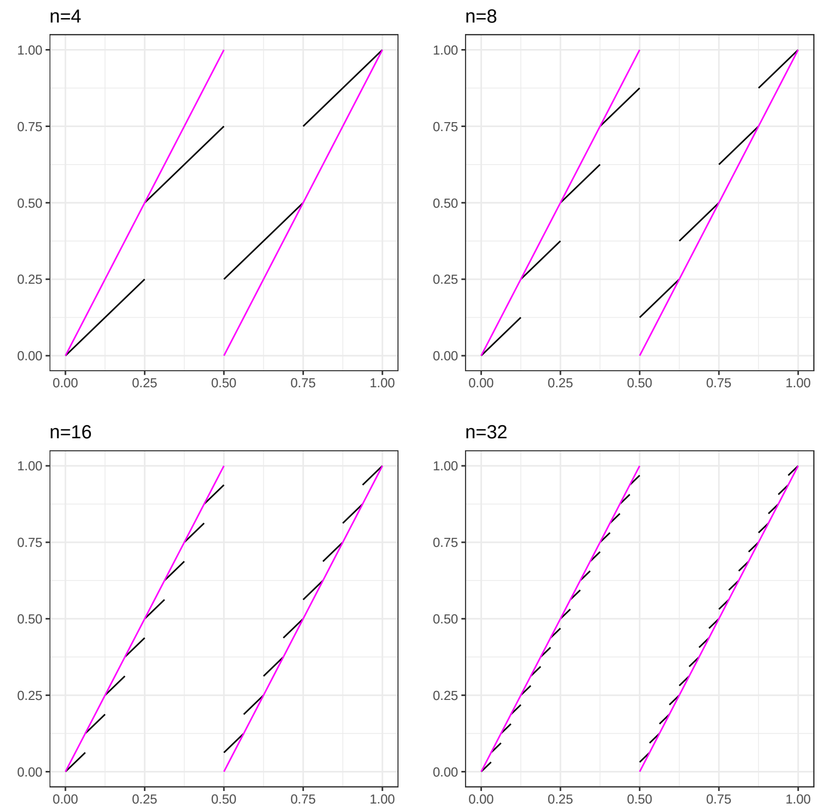

Figure 1: The λ-preserving transformations S_n (black) for $n \in \{4, 8, 16, 32\}$ and S (magenta), defined in Example 3.3.

$\mathcal{S}_{(S_1 \circ S_2 \times T_1 \circ T_2)}(A) = \mathcal{S}_{(S_1 \times T_1)}(\mathcal{S}_{(S_2 \times T_2)}(A))$ holds for every $A \in \mathcal{C}$ the desired result now follows from the fact that the family of all shuffles of M are dense in $(\mathcal{C}, d_\infty)$ (see [8, 18]). The case that C is right-invertible can be handled analogously. $\square$

In the next section we study double-shuffles induced by strongly mixing transformations T_1 and T_2 and derive some properties of copulas invariant with respect to double-shuffles of this kind.

4. Copulas invariant with respect to double shuffles induced by strongly mixing transformations

In the sequel, given $T_1, T_2 \in \mathcal{T}$ we will write $\boldsymbol{T} := (T_1 \times T_2) \colon [0,1]^2 \to [0,1]^2$ and study the family of all $\mathcal{S}_{\boldsymbol{T}}$-invariant copulas, defined by

$$\Omega_{\boldsymbol{T}} := \{A \in \mathcal{C} \colon \mathcal{S}_{\boldsymbol{T}}(A) = A\}. \tag{10}$$

Each copula $A \in \Omega_{\boldsymbol{T}}$ will be called $\mathcal{S}_{\boldsymbol{T}}$-invariant. We start with a simple first result which will be refined for special classes of transformations T_1, T_2 throughout the rest of the paper.

Theorem 4.1. *The family $\Omega_{\boldsymbol{T}}$ is a convex and compact subset of $(\mathcal{C}, d_\infty)$.*

Proof. Convexity is obvious, to prove compactness we may proceed as follows: Considering that $(\mathcal{C}, d_\infty)$ is compact it suffices to prove that $\Omega_{\boldsymbol{T}}$ is closed. Letting $(A_n)_{n\in\mathbb{N}}$ denote a sequence of $\mathcal{S}_{\boldsymbol{T}}$-invariant copulas with d_∞-limit $A \in \mathcal{C}$ according to Lemma 3.2 we have

$$d_\infty(A, \mathcal{S}_{\boldsymbol{T}}(A)) = \lim_{n\to\infty} d_\infty(A_n, \mathcal{S}_{\boldsymbol{T}}(A)) = \lim_{n\to\infty} d_\infty(\mathcal{S}_{\boldsymbol{T}}(A_n), \mathcal{S}_{\boldsymbol{T}}(A)) = 0,$$

so $A \in \Omega_{\boldsymbol{T}}$. $\square$

Remark 4.2. Lemma 3.2 also yields completeness of $(\Omega_{\boldsymbol{T}}, D_1)$. In fact, considering that $(\mathcal{C}, D_1)$ is complete and that convergence w.r.t. D_1 implies convergence w.r.t. d_∞ Theorem 4.1 immediately yields completeness. However, as illustrated by the subsequent example, $\Omega_{\boldsymbol{T}}$ is not compact in $(\mathcal{C}, D_1)$.

Example 4.3. Let $(A_{h_n})_{n\in 2\mathbb{N}}$ be a sequence of completely dependent copulas, whereby the λ-preserving transformations $h_n \colon [0,1] \to [0,1]$ are defined by

$$h_n(x) := 2nx \,(mod 1)$$

for every $n \in \mathbb{N}$ and $x \in [0,1]$ (see Figure 2).
Considering $D_1(A^t_{h_n}, \Pi) \leq \frac{1}{2n}$ it follows immediately that $(A^t_{h_n})_{n\in\mathbb{N}}$ converges to Π w.r.t. d_∞ and, using the fact that $d_\infty(A^t, B^t) = d_\infty(A, B)$ holds for all $A, B \in \mathcal{C}$, the same is true for $(A_{h_n})_{n\in\mathbb{N}}$. Define the transformations $T_1, T_2 \in \mathcal{T}$ by $T = T_1 = T_2 := h_1$. It is well known that T is isomorphic to a Bernoulli shift (see [3, 28]) and hence strongly mixing. Using disintegration and change of coordinates we get

$$\mathcal{S}_{\boldsymbol{T}}(A_{h_n})(x, y) = \int_{T^{-1}([0,x])} \mathbb{1}_{T^{-1}([0,y])} \circ h_n(t) d\lambda(t)$$

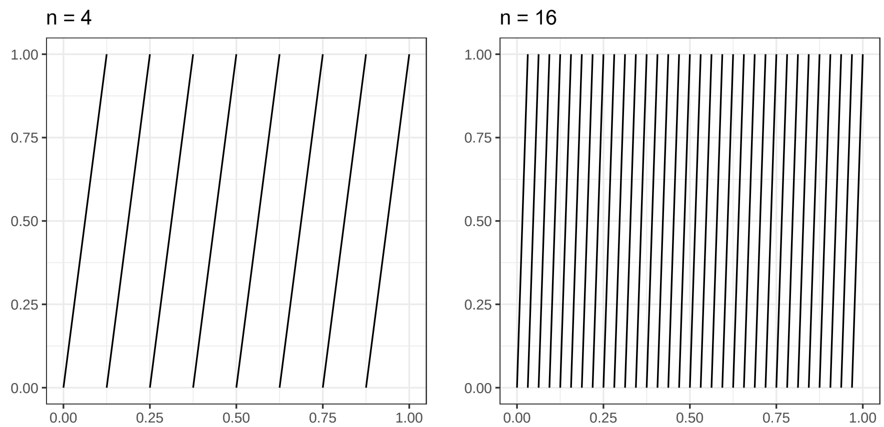

Figure 2: The λ-preserving transformations h_n for $n \in \{4, 16\}$ considered Example 4.3.

$$
\begin{aligned}
&= \int_{[0,1]} \big(\mathbb{1}_{[0,x]} \circ T(t)\big) \cdot \big(\mathbb{1}_{[0,y]} \circ T \circ h_n(t)\big)\, d\lambda(t) \\
&= \int_{[0,1]} \big(\mathbb{1}_{[0,x]} \circ T(t)\big) \cdot \big(\mathbb{1}_{[0,y]} \circ h_n \circ T(t)\big)\, d\lambda(t) \\
&= \int_{[0,1]} \mathbb{1}_{[0,x]}(t) \cdot \big(\mathbb{1}_{[0,y]} \circ h_n(t)\big)\, d\lambda(t) = A_{h_n}(x, y).
\end{aligned}
$$

for all $x, y \in [0,1]$, i.e., $A_{h_n} \in \Omega_{\boldsymbol{T}}$. If $(\Omega_{\boldsymbol{T}}, D_1)$ were compact then we could find a D_1-convergent subsequence $\left(A_{h_{n_j}}\right)_{j\in\mathbb{N}}$ with limit A_h for some $h \in \mathcal{T}$. Again using the fact that D_1-convergence implies d_∞-convergence and the afore-mentioned d_∞-invariance with respect to transposition the property $A_h = \Pi$ would follow. The latter, however, is impossible since A_h is singular and Π is absolutely continuous (see Theorem 14 in [21]).

In the sequel we assume that $T_1, T_2 \in \mathcal{T}$ are strongly mixing transformations on $([0,1], \mathcal{B}([0,1]), \lambda)$. It is easy to verify (see for instance [2]) that in this case the map $\boldsymbol{T} = (T_1 \times T_2)\colon [0,1]^2 \to [0,1]^2$ is strongly mixing on the product space $([0,1]^2, \mathcal{B}([0,1]^2), \lambda_2)$. The following theorem states that every $\mathcal{S}_{\boldsymbol{T}}$-invariant copula A exhibiting an absolute continuous component can be expressed as convex combination of the product-copula Π and a (purely) singular $\mathcal{S}_{\boldsymbol{T}}$-invariant copula.

Theorem 4.4. *Suppose that $T_1, T_2 \in \mathcal{T}$ are strongly mixing and let $A \in \mathcal{C}$ be $\mathcal{S}_{\boldsymbol{T}}$-invariant with $m_A^{\ll} > 0$. If $m_A^{\ll} = 1$ then $A = \Pi$, otherwise there* *exists a singular $\mathcal{S}_{\boldsymbol{T}}$-invariant copula $B \in \mathcal{C}$ fulfilling*

$$A = m_A^{\ll}\Pi + (1 - m_A^{\ll})B.$$

Proof. Suppose that $A \in \mathcal{C}$ be $\mathcal{S}_{\boldsymbol{T}}$-invariant. Then for every $G \in \mathcal{B}([0,1]^2)$ and every $n \in \mathbb{N}$ we have

$$\begin{aligned}
\mu_A(G) &= \mu_{\mathcal{S}_{\boldsymbol{T}}(A)}(G) = \mu_A(\boldsymbol{T}^{-n}(G)) = \mu_A^{\perp}(\boldsymbol{T}^{-n}(G)) + \mu_A^{\ll}(\boldsymbol{T}^{-n}(G)) \\
&= \mu_A^{\perp}(\boldsymbol{T}^{-n}(G)) + \underbrace{\int_{[0,1]^2} k_A(x,y)\,(\mathbb{1}_G \circ \boldsymbol{T}^n(x,y))\; d\lambda_2(x,y)}_{=:I_n},
\end{aligned}$$

where $k_A(x,y)$ denotes (a version of) the Radon-Nikodym density of $\mu_A^{\ll}$. Applying Lemma 2.1 yields $\lim_{n\to\infty} I_n = m_A^{\ll} \cdot \lambda_2(G)$. In the case that $m_A^{\ll} = 1$ we have $\mu_A(G) = \lambda_2(G)$ for every $G \in \mathcal{B}([0,1]^2)$, hence $A = \Pi$. In case of $m_A^{\ll} \in (0,1)$ the sequence $\big(\mu_A^{\perp}(\boldsymbol{T}^{-n}(G))\big)_{n\in\mathbb{N}}$ converges to $\mu_A(G) - m_A^{\ll}\lambda_2(G)$ and we can proceed as follows.
Defining $\vartheta(G) := \lim_{n\to\infty} \mu_A^{\perp}(\boldsymbol{T}^{-n}(G))$ for every $G \in \mathcal{B}([0,1]^2)$ we obtain a non-negative measure on $\mathcal{B}([0,1]^2)$ fulfilling $\vartheta([0,1]^2) = 1 - m_A^{\ll}$. Setting $\mu_B := \frac{\vartheta}{1-m_A^{\ll}}$ it follows that μ_B is doubly stochastic. Since $\mathcal{S}_{\boldsymbol{T}}$-invariance of B is a direct consequence of the fact that A and Π are $\mathcal{S}_{\boldsymbol{T}}$-invariant it remains to show that μ_B is singular w.r.t. λ_2. Choose $G \in \mathcal{B}([0,1]^2)$ such that $\lambda_2(G) = 1$ and $\mu_A^{\perp}(G) = 0$. Such a set $G \neq [0,1]^2$ exists since $m_A^{\ll} < 1$ by assumption. Altogether we have

$$\begin{aligned}
\mu_B(G) &= \frac{\vartheta(G)}{1 - m_A^{\ll}} = \frac{\mu_A(G) - m_A^{\ll}\lambda_2(G)}{1 - m_A^{\ll}} = \frac{\mu_A^{\ll}(G) + \mu_A^{\perp}(G) - m_A^{\ll}}{1 - m_A^{\ll}} \\
&= \frac{\mu_A^{\ll}(G) - m_A^{\ll}}{1 - m_A^{\ll}} \leq \frac{\mu_A^{\ll}([0,1]^2) - m_A^{\ll}}{1 - m_A^{\ll}} = 0,
\end{aligned}$$

which completes the proof. □

Remark 4.5. Theorem 4.4 states that the product copula Π is the only absolutely continuous $\mathcal{S}_{\boldsymbol{T}}$-invariant copula for strongly mixing transformations T_1, T_2. Note that Theorem 3.2 in [11] is a special case of Theorem 4.4.

Theorem 4.6. *Suppose that $T_1, T_2 \in \mathcal{T}$ are strongly mixing. Then the family $\Omega_{\boldsymbol{T}}$ is a convex and compact subset of $(\mathcal{C}, d_\infty)$ and the product copula Π is an extreme point of $\Omega_{\boldsymbol{T}}$.*

Proof. Suppose that $\alpha \in (0,1)$, that $A, B \in \Omega_{\boldsymbol{T}}$ and that $\Pi = \alpha A + (1-\alpha)B$ holds. Then μ_A, μ_B are absolutely continuous w.r.t. $\mu_\Pi = \lambda_2$, so Theorem 4.4 implies $A = B = \Pi$, which shows that Π is extremal in $\Omega_{\boldsymbol{T}}$. □

Deriving a full analytic characterization of all elements or of all extreme points of $\Omega_{\boldsymbol{T}}$ for arbitrary strongly mixing transformations $T_1, T_2 \in \mathcal{T}$ seems intractable. For special transformations, however, various results can be derived. In [11] Horanská and Sarkoci constructed numerous $\mathcal{S}_{\boldsymbol{T}}$-invariant copulas for the case that T_1, T_2 are so-called tent-maps. In the sequel we show that the essential property of tent maps assuring the construction of various $\mathcal{S}_{\boldsymbol{T}}$-invariant copulas as done in [11] is that each transformation maps each element of a finite partition of $[0,1]$ into intervals linearly onto $[0,1]$. Since this very property is also fulfilled by all so-called Lüroth-maps we will work with those transformations in the sequel and construct among other things uncountably many extreme points of $\Omega_{\boldsymbol{T}}$.

5. Copulas invariant with respect to Lüroth double shuffles and related results

Before focusing on Lüroth double shuffles we will recall the definition of Iterated Function Systems with Probabilities (IFSP) and Lüroth maps. Following [1, 15, 16] we can summarize the basic construction as follows: Let (Ω, d) be a complete metric space. A function $\omega\colon \Omega \to \Omega$ is a contraction if there exists $L < 1$ such that $d(\omega(x), \omega(y)) \leq L \cdot d(x,y)$ for all $x, y \in \Omega$. A family of contractions $\{\omega_1, \ldots, \omega_n\}$ with $n \geq 2$ is called an Iterated Function System (IFS) and we will denote it by $\{\Omega, (\omega_i)_{i=1}^n\}$. An IFS together with probabilities $(p_i)_{i=1}^n$ with $p_i \geq 0$ for all $i = 1, \ldots, n$ and $\sum_{i=1}^n p_i = 1$ is called an Iterated Function System with Probabilities (IFSP) and is denoted by $\{\Omega, (\omega_i)_{i=1}^n, (p_i)_{i=1}^n\}$. Every IFSP induces the so-called Hutchinson operator $\mathcal{H}$, defined by

$$\mathcal{H}\colon \mathcal{K}(\Omega) \to \mathcal{K}(\Omega) \text{ with } \mathcal{H}(K) = \bigcup_{\substack{i=1 \\ p_i > 0}}^{n} w_i(K),$$

where $\mathcal{K}(\Omega)$ denotes the family of all nonempty, compact subsets of Ω. The map $\mathcal{H}$ is a contraction w.r.t. the Hausdorff-metric δ_H on $\mathcal{K}(\Omega)$. According to Banach's Fixed Point Theorem there exists a unique $K^\star \in \mathcal{K}(\Omega)$ with $\mathcal{H}(K^\star) = K^\star$ such that $\lim_{n\to\infty} \delta_H(\mathcal{H}^n(K), K^\star) = 0$ holds for every $K \in \mathcal{K}(\Omega)$. $K^\star$ is called the attractor of the IFSP. Furthermore letting $\mathcal{P}(\Omega)$ denote the family of all probability measures on $\mathcal{B}(\Omega)$, each IFSP induces a so-called Markov operator $\mathcal{V}\colon \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$, defined by

$$\mathcal{V}(\mu) := \sum_{i=1}^{n} p_i \mu^{\omega_i},$$

where μ^{ω_i} denotes the push-forward of μ via ω_i. Notice that this operator has nothing to do with the Markov operators on $L^1([0,1],\mathcal{B}([0,1]),\lambda)$ that are in one-to-one correspondence with the class of bivariate copulas (see [5, 21]). We nevertheless stick to the name Markov operator since it is common in the literature and since no confusion will arise in the sequel. It can be shown that $\mathcal{V}$ is a contraction on $(\mathcal{P}(\Omega),\delta_K)$, where δ_K denotes the Kantorovich metric (see [22]), defined by

$$\delta_K(\mu,\nu) := \sup\left\{\int_\Omega f\,d\mu - \int_\Omega f\,d\nu \colon f \in Lip_1(\Omega,\mathbb{R})\right\}$$

and $Lip_1(\Omega,\mathbb{R}) := \{h\colon \Omega \to \mathbb{R}\colon |h(x)-h(y)| \le d(x,y) \text{ for all } x,y \in \Omega\}$. Again by Banach's Fixed Point Theorem there exists a unique $\mu^\star \in \mathcal{P}(\Omega)$ fulfilling $\mathcal{V}(\mu^\star) = \mu^\star$ and

$$\lim_{n\to\infty} \delta_K(\mathcal{V}^n(\mu),\mu^\star) = 0$$

holds for all $\mu \in \mathcal{P}(\Omega)$. Moreover the attractor $K^\star$ is the support of the unique invariant measure $\mu^\star$. For more background information on the Markov and the Hutchinson operator see [1, 15, 16] and the references therein.

In the sequel we focus on double shuffle induced by pairs of so-called finite Lüroth maps (where finite refers to the cardinality of the underlying partition of $[0,1]$). Let $\{I_1,I_2,\dots,I_n\}$ denote a partition of the unit interval for $0 \le n \in \mathbb{N}$, induced by a strictly increasing vector $\boldsymbol{a} = (a_0,a_1,\dots,a_n) \in [0,1]^{n+1}$ fulfilling $a_0 = 0$, $a_n = 1$, i.e., we have $I_1 := [a_0,a_1]$, $I_2 := (a_1,a_2]$, $\dots$, $I_n = (a_{n-1},a_n]$. For each such partition the $\boldsymbol{a}$-Lüroth map $T_{\boldsymbol{a}}\colon [0,1] \to [0,1]$ is defined by

$$T_{\boldsymbol{a}}(x) = \tfrac{x-a_{j-1}}{a_j-a_{j-1}} \text{ for } x \in I_j \text{ and } j \in \{1,2,\dots,n\}.$$

Figure 3 depicts some examples of $\boldsymbol{a}$-Lüroth maps. Obviously each Lüroth map $T_{\boldsymbol{a}}$ is λ-preserving. Furthermore $T_{\boldsymbol{a}}$ is isomorphic to the shift operator σ on the code space $\Sigma_n := \{1,\dots,n\}^{\mathbb{N}}$ endowed with the product measure $\mathbb{P}_{\boldsymbol{a}}$ induced by the point probabilities $p_i = \lambda(I_i), i \in \{1,\dots,n\}$. In other words, the dynamical system $([0,1],\mathcal{B}([0,1]),\lambda,T_{\boldsymbol{a}})$ is a Bernoulli shift (see [3, 28]), hence strongly mixing. In the sequel we will write

$$\boldsymbol{T}_{\boldsymbol{a},\boldsymbol{b}}(x,y) := (T_{\boldsymbol{a}} \times T_{\boldsymbol{b}})(x,y) = (T_{\boldsymbol{a}}(x),T_{\boldsymbol{b}}(b))$$

and refer to $\boldsymbol{T}_{\boldsymbol{a},\boldsymbol{b}}\colon [0,1]^2 \to [0,1]^2$ as bivariate Lüroth map (corresponding to $\boldsymbol{a},\boldsymbol{b}$) and to $\mathcal{S}_{\boldsymbol{T}_{\boldsymbol{a},\boldsymbol{b}}}$ as Lüroth double shuffle.

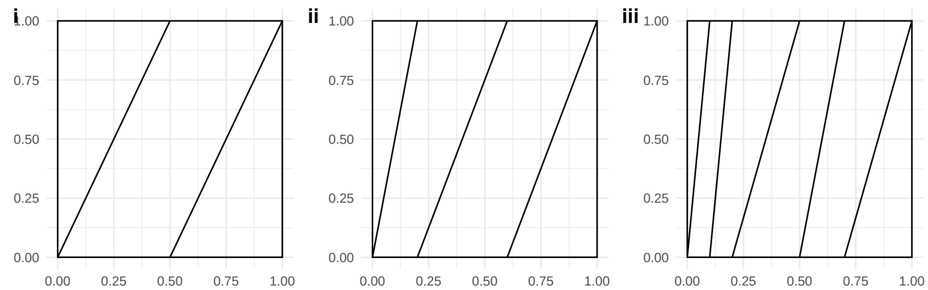

Figure 3: Three examples of $\boldsymbol{a}$-Lüroth maps $T_{\boldsymbol{a}}$ corresponding to $\boldsymbol{a} = (0, 0.5, 1)$ (first panel), $\boldsymbol{a} = (0, 0.2, 0.6, 1)$ (second panel) and $\boldsymbol{a} = (0, 0.1, 0.2, 0.5, 0.7, 1)$ (third panel).

Following the ideas from [11] we are now going to study the family of all $\mathcal{S}_{\boldsymbol{T}_{\boldsymbol{a},\boldsymbol{b}}}$-invariant copulas and derive results analogous to the ones from [11] as well as various novel properties.

Remark 5.1. As already mentioned, in [11] double shuffles induced by tent maps (corresponding to partitions of $[0,1]$ into two subintervals) and not induced by Lüroth maps are studied. Nevertheless our results imply the ones in [11]. In fact, considering $\boldsymbol{a} = (0, a_1, 1), \boldsymbol{b} = (0, b_1, 1)$ and letting

$$t_{\boldsymbol{a},\boldsymbol{b}} = (t_{\boldsymbol{a}} \times t_{\boldsymbol{b}})\colon [0,1]^2 \to [0,1]^2$$

denote the bivariate tent map composed by the univariate tent maps $t_{\boldsymbol{a}}$ and $t_{\boldsymbol{b}}\colon [0,1] \to [0,1]$ it is straightforward to verify that there is a one-to-one correspondence between $\mathcal{S}_{T_{\boldsymbol{a},\boldsymbol{b}}}$-invariant and $\mathcal{S}_{t_{\boldsymbol{a},\boldsymbol{b}}}$-invariant copulas (direct consequence of the fact that the dynamical systems $([0,1]^2, \mathcal{B}([0,1]^2, \lambda_2, T_{\boldsymbol{a},\boldsymbol{b}})$ and $([0,1]^2, \mathcal{B}([0,1]^2, \lambda_2, t_{\boldsymbol{a},\boldsymbol{b}})$ are isomorphic since both are isomorphic to the same Bernoulli shift). We chose to work with Lüroth maps since assuming that the maps are increasing on each interval keeps notation simple.

Theorem 4.4 implies that Π is the only absolutely continuous copula invariant with respect to the double shuffle $\mathcal{S}_{\boldsymbol{T}_{\boldsymbol{a},\boldsymbol{b}}} =: \mathcal{S}_{\boldsymbol{a},\boldsymbol{b}}$. Dropping absolute continuity opens the door for constructing rich families of $\mathcal{S}_{\boldsymbol{a},\boldsymbol{b}}$-invariant copulas. Following [11] we first construct a transformation which acts as right inverse of the double shuffle $\mathcal{S}_{\boldsymbol{a},\boldsymbol{b}}$. To simplify notation we work with transformation matrices introduced in [10] (also see [22]).

Definition 5.2. A $k \times l$ matrix $T = (t_{ij})_{i=1,\dots,k, j=1,\dots,l}$ is called transformation matrix if it fulfills i) $k, l \geq 2$, ii) all entries are non negative, iii) $\sum_{i=1}^{k} \sum_{j=1}^{l} t_{ij} = 1$ and iv) no row or column has all entries 0.

Let $R_{ij} = [a_{i-1}, a_i] \times [b_{j-1}, b_j]$ for $1 \le i \le k$ and $1 \le j \le l$ be rectangles in $[0,1]^2$, whereby $\bigcup_{i,j=1}^{k,l} R_{ij} = [0,1]^2$, and $\omega_{ij}\colon [0,1]^2 \to R_{ij}$ are defined by

$$\omega_{ij}(x,y) \coloneqq (a_{i-1} + x(a_i - a_{i-1}), b_{j-1} + y(b_j - b_{j-1})),$$

for all $i \in \{1,2,\dots,k\}$ and $j \in \{1,2,\dots,l\}$.
If a transformation matrix $\boldsymbol{\theta} \coloneqq (\theta_{ij}) \in [0,1]^{k\times l}$ satisfies the properties

1. $\forall i \in \{1,\dots,k\}\colon\ \sum_{j=1}^{l} \theta_{ij} = a_i - a_{i-1}$
2. $\forall j \in \{1,\dots,l\}\colon\ \sum_{i=1}^{k} \theta_{ij} = b_j - b_{j-1}$

then we will refer to $\boldsymbol{\theta}$ as *compatible* with $T_{\boldsymbol{a},\boldsymbol{b}}$.

Remark 5.3. It is straightforward to verify (see Definition 5 and the subsequent paragraph in [24]) that for each $T_{\boldsymbol{a},\boldsymbol{b}}$ we can find uncountably many compatible transformation matrices. In fact, it is possible to construct uncountably many compatible transformation matrices consisting of strictly positive entries and, in the case that $\boldsymbol{a}$ or $\boldsymbol{b}$ has length ≥ 4 uncountably many containing at least one 0.

Example 5.4. Consider the $\boldsymbol{a}$-Lüroth maps $T_{\boldsymbol{a}}$ with $\boldsymbol{a} = (0, 0.3, 1)$ and $T_{\boldsymbol{b}}$ with $\boldsymbol{b} = (0, 0.25, 0.75, 1)$. Then two compatible transformation matrices are given by

$$\boldsymbol{\theta}_1 = \begin{pmatrix} 0 & 0.3 & 0 \\ 0.25 & 0.2 & 0.25 \end{pmatrix}, \quad \boldsymbol{\theta}_2 = \begin{pmatrix} 0.1 & 0.1 & 0.1 \\ 0.15 & 0.4 & 0.15 \end{pmatrix}.$$

To simplify notation, in the sequel we will write $\tilde{I} \coloneqq \{(i,j)\colon \theta_{ij} > 0\}$ and consider the IFSP $\{[0,1]^2, (\omega_{ij})_{(i,j)\in\tilde{I}}, (\theta_{ij})_{(i,j)\in\tilde{I}}\}$. The induced Markov operator $\mathcal{V}^{\boldsymbol{\theta}}_{\boldsymbol{a},\boldsymbol{b}}$ is then defined by

$$\mathcal{V}^{\boldsymbol{\theta}}_{\boldsymbol{a},\boldsymbol{b}}(\mu_A) \coloneqq \sum_{i=1}^{k}\sum_{j=1}^{l} \theta_{ij}\mu_A^{\omega_{ij}} = \sum_{(i,j)\in\tilde{I}} \theta_{ij}\mu_A^{\omega_{ij}} \tag{11}$$

for all doubly stochastic measures μ_A. Since $\mathcal{V}^{\boldsymbol{\theta}}_{\boldsymbol{a},\boldsymbol{b}}$ maps doubly stochastic measures into itself if $\boldsymbol{\theta}$ is compatible with $T_{\boldsymbol{a},\boldsymbol{b}}$ (if not, $\mathcal{V}^{\boldsymbol{\theta}}_{\boldsymbol{a},\boldsymbol{b}}(\mu_A)$ is not necessarily doubly stochastic since the affine transformations w_{ij} are defined according to $\boldsymbol{a}, \boldsymbol{b}$) we may view $\mathcal{V}^{\boldsymbol{\theta}}_{\boldsymbol{a},\boldsymbol{b}}$ also as operator on $\mathcal{C}$. Considering that $\mathcal{V}^{\boldsymbol{\theta}}_{\boldsymbol{a},\boldsymbol{b}}$ is a contraction on the complete metric space $(\mathcal{C}, D_1)$ the following proposition follows (see [21, 24]):

Proposition 5.5. *Let* $\mathcal{V}^{\boldsymbol{\theta}}_{\mathbf{a},\boldsymbol{b}}$ *be defined according to equation* (11). *Then there exists a unique copula* $C^{\star}_{\boldsymbol{\theta}}$ *fulfilling* $\mathcal{V}^{\boldsymbol{\theta}}_{\mathbf{a},\boldsymbol{b}}(C^{\star}_{\boldsymbol{\theta}}) = C^{\star}_{\boldsymbol{\theta}}$. *Furthermore for every copula* C *we have*

$$\lim_{n\to\infty} D_1\left(\left(\mathcal{V}^{\boldsymbol{\theta}}_{\mathbf{a},\boldsymbol{b}}\right)^n(C), C^{\star}_{\boldsymbol{\theta}}\right) = 0.$$

The following lemma shows that $\mathcal{V}^{\boldsymbol{\theta}}_{\mathbf{a},\boldsymbol{b}}$ works as right-inverse of $\mathcal{S}_{\boldsymbol{a},\boldsymbol{b}}$ and is therefore of interest for finding $\mathcal{S}_{\boldsymbol{a},\boldsymbol{b}}$-invariant copulas.

Lemma 5.6. *Let* $\mathcal{V}^{\boldsymbol{\theta}}_{\mathbf{a},\boldsymbol{b}}$ *be defined according to equation* (11). *Then for every copula* $A \in \mathcal{C}$ *we have* $\mathcal{S}_{\boldsymbol{a},\boldsymbol{b}}(\mathcal{V}^{\boldsymbol{\theta}}_{\mathbf{a},\boldsymbol{b}}(\mu_A)) = \mu_A$.

Proof. It suffices to show $\left(\mathcal{V}^{\boldsymbol{\theta}}_{\mathbf{a},\boldsymbol{b}}(\mu_A)\right)^{\boldsymbol{T}_{\boldsymbol{a},\boldsymbol{b}}}(E\times F) = \mu_A(E\times F)$ for all $E, F \in \mathcal{B}([0,1])$ and arbitrary $A \in \mathcal{C}$. The latter, however is a direct consequence, of the following straightforward calculation:

$$\begin{aligned}
\left(\mathcal{V}^{\boldsymbol{\theta}}_{\mathbf{a},\boldsymbol{b}}(\mu_A)\right)^{\boldsymbol{T}_{\boldsymbol{a},\boldsymbol{b}}}(E\times F) &= \left(\sum_{(i,j)\in\tilde{I}} \theta_{ij}\mu_A^{\omega_{ij}}\right)^{\boldsymbol{T}_{\boldsymbol{a},\boldsymbol{b}}}(E\times F)\\
&= \sum_{(i,j)\in\tilde{I}} \theta_{ij}\mu_A^{\omega_{ij}}(\boldsymbol{T}^{-1}_{\boldsymbol{a},\boldsymbol{b}}(E\times F))\\
&= \sum_{(i,j)\in\tilde{I}} \theta_{ij}\mu_A\left(\omega_{ij}^{-1}(\boldsymbol{T}^{-1}_{\boldsymbol{a},\boldsymbol{b}}(E\times F))\right)\\
&= \sum_{(i,j)\in\tilde{I}} \theta_{ij}\mu_A\left(\omega_{ij}^{-1}(T^{-1}_{\boldsymbol{a}}(E)\times T^{-1}_{\boldsymbol{b}}(F))\right)\\
&= \sum_{(i,j)\in\tilde{I}} \theta_{ij}\mu_A\left(\omega_{ij}^{-1}\left(T^{-1}_{\boldsymbol{a}}(E)\times T^{-1}_{\boldsymbol{b}}(F)\cap R_{i,j}\right)\right)\\
&= \sum_{(i,j)\in\tilde{I}} \theta_{ij}\mu_A(E\times F) = \mu_A(E\times F),
\end{aligned}$$

□

Combining Proposition 5.5 and Lemma 5.6 we obtain the following very useful implication for copulas $C \in \mathcal{C}$ and compatible transformation matrices $\boldsymbol{\theta}$ already established by Horanská and Sarkoci in [11] in the context of tent maps:

$$\mathcal{V}^{\boldsymbol{\theta}}_{\mathbf{a},\boldsymbol{b}}(\mu_C) = \mu_C \Longrightarrow \mathcal{S}_{\boldsymbol{a},\boldsymbol{b}}(C) = C \tag{12}$$

The next proposition is the Lüroth version of Proposition 3.3. in [11]:

Proposition 5.7. *Let $\mathcal{V}^{\boldsymbol{\theta}}_{\boldsymbol{a},\boldsymbol{b}}$ be defined according to equation (11). Then the unique copula $C^\star_{\boldsymbol{\theta}}$ from Proposition 5.5 is $\mathcal{S}_{\boldsymbol{a},\boldsymbol{b}}$-invariant. Moreover, the following assertions hold:*

1. *If $\theta_{ij} = (a_i - a_{i-1})(b_j - b_{j-1})$ for all $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, l\}$ then $C^\star_{\boldsymbol{\theta}} = \Pi$.*

2. *If $\theta_{ij} > 0$ for all i, j and if there exists at least one (i, j) such that $\theta_{ij} \neq (a_i - a_{i-1})(b_j - b_{j-1})$ then $C^\star_{\boldsymbol{\theta}}$ is singular and has full support.*

3. *If at least one entry of $\boldsymbol{\theta}$ is zero then $C^\star_{\boldsymbol{\theta}}$ is singular and the support $K^\star_{\boldsymbol{\theta}}$ of $C^\star_{\boldsymbol{\theta}}$ fulfills $\lambda_2(K^\star_{\boldsymbol{\theta}}) = 0$.*

Proof. The invariance property of $C^\star_{\boldsymbol{\theta}}$ is a direct consequence of equation (12). The first assertion is a direct consequence of $\mathcal{V}^{\boldsymbol{\theta}}_{\boldsymbol{a},\boldsymbol{b}}(\mu_\Pi) = \mu_\Pi$. Considering the fact that the support of the invariant copula $C^\star_{\boldsymbol{\theta}}$ coincides with the corresponding fixed point of the Hutchinson operator $\mathcal{H}^{\boldsymbol{\theta}}_{\boldsymbol{a},\boldsymbol{b}}$ the second assertion follows from the fact that

$$\mathcal{H}^{\boldsymbol{\theta}}_{\boldsymbol{a},\boldsymbol{b}}([0,1]^2) = \bigcup_{(i,j)\in I} \omega_{ij}([0,1]^2) = [0,1]^2.$$

Hence the support of $C^\star_{\boldsymbol{\theta}}$ is $[0,1]^2$ and, using the results in [24], singularity of $C^\star_{\boldsymbol{\theta}}$ follows. In the third situation $\lambda_2(K^\star_{\boldsymbol{\theta}}) = 0$ is easily verified. Having this, singularity of $C^\star_{\boldsymbol{\theta}}$ is trivial. □

As next step we take a look at the entropy of the dynamical system $(K^\star_{\boldsymbol{\theta}}, \mathcal{B}(K^\star_{\boldsymbol{\theta}}), \mu_{C^\star_{\boldsymbol{\theta}}}, \boldsymbol{T}_{\boldsymbol{a},\boldsymbol{b}})$ and start with a direct consequence of Theorem 3 in [24]:

Corollary 5.8. *The dynamical system $(K^\star_{\boldsymbol{\theta}}, \mathcal{B}(K^\star_{\boldsymbol{\theta}}), \mu_{C^\star_{\boldsymbol{\theta}}}, \boldsymbol{T}_{\boldsymbol{a},\boldsymbol{b}})$ is Bernoulli, hence strongly mixing, and its entropy $h_{\mu_{C^\star_{\boldsymbol{\theta}}}}(\boldsymbol{T}_{\boldsymbol{a},\boldsymbol{b}})$ is given by*

$$h_{\mu_{C^\star_{\boldsymbol{\theta}}}}(\boldsymbol{T}_{\boldsymbol{a},\boldsymbol{b}}) = -\sum_{(i,j)\in\tilde{I}} \theta_{ij} \cdot log(\theta_{ij}).$$

Obviously, given an arbitrary $s \in (0, \infty)$ it is always possible to find $\boldsymbol{a}, \boldsymbol{b}$ and a transformation matrix $\boldsymbol{\theta}$ such that the corresponding $\mathcal{S}_{T_{\boldsymbol{a},\boldsymbol{b}}}$-invariant copula $C^\star_{\boldsymbol{\theta}}$ has entropy s. On the other hand, given a 2-dimensional Lüroth map $\boldsymbol{T}_{\boldsymbol{a},\boldsymbol{b}}$ it does not seem straightforward to determine $h_{\mu_{C^\star_{\boldsymbol{\theta}}}}(\boldsymbol{T}_{\boldsymbol{a},\boldsymbol{b}})$ for all compatible transformation matrices $\boldsymbol{\theta}$. The following remark provides a sharp upper bound and a lower bound which may not be attained except in special cases.

Remark 5.9. Let $\boldsymbol{T_{a,b}}$ be an arbitrary 2-dimensional Lüroth map. Then setting $\overline{a}_i = a_i - a_{i-1}$ and $\overline{b}_j = b_j - b_{j-1}$ we have

$$h_{\mu_{C^\star_{\boldsymbol{\theta}}}}(\boldsymbol{T_{a,b}}) \in \left[-\sum_{j=1}^{l} \max_i(\theta_{ij}\} \cdot \log(\max_i(\theta_{ij})), - \sum_{(i,j)\in\tilde{I}} \overline{a}_i\overline{b}_j \cdot \log(\overline{a}_i\overline{b}_j) \right].$$

It is straightforward to verify that the upper bound is attained by considering the transformation matrix $\theta_{i,j} = \overline{a}_i\overline{b}_j, i \in \{1, \ldots, k\}, j \in \{1, \ldots, l\}$, in which case we have $C^\star_{\boldsymbol{\theta}} = \Pi$. The lower bound is obvious, and it is attained if the invariant copula $C^\star_{\boldsymbol{\theta}}$ is mutually completely dependent (see [21] for properties of completely dependent copulas). Notice that the latter can only be the case if there exists a permutation τ of $\{1, \ldots, k\}$ such that for all $i \in \{1, \ldots, k\}$ we have $a_i - a_{i-1} = b_{\tau(i)} - b_{\tau(i)-1}$.

In the following examples we illustrate Proposition 5.7 and Corollary 5.8 and plot the density of the probability measure $\left(\mathcal{V}^{\boldsymbol{\theta}}_{\boldsymbol{a,b}}\right)^n(\Pi)$ for various choices of $\boldsymbol{a}$ and $\boldsymbol{b}$ and compatible transformation matrices $\boldsymbol{\theta}$. In the setting considered in Example 5.10 the invariant copula has full support in Example 5.11 and Example 5.12 the support of the copulas is a set of λ_2-measure 0.

Example 5.10. Set $\boldsymbol{a} = (0, 0.1, 0.4, 0.8, 1)$, $\boldsymbol{b} = (0, 0.3, 0.7, 1)$ and consider the compatible transformation matrix $\boldsymbol{\theta}$ given by

$$\boldsymbol{\theta} = \begin{pmatrix} 0.02 & 0.03 & 0.05 \\ 0.07 & 0.14 & 0.09 \\ 0.18 & 0.15 & 0.07 \\ 0.03 & 0.08 & 0.09 \end{pmatrix}.$$

Figure 4 depicts the Lüroth-maps $T_{\boldsymbol{a}}$ and $T_{\boldsymbol{b}}$, respectively, Figure 5 depicts the densities of $\left(\mathcal{V}^{\boldsymbol{\theta}}_{\boldsymbol{a,b}}\right)^n(\Pi)$ for $n \in \{1, 2, 3, 5\}$. According to Proposition 5.7 the invariant copula $C^\star_{\boldsymbol{\theta}}$ is singular and has full support. Moreover, the entropy of $\boldsymbol{T_{a,b}}$ w.r.t. to $\mu_{C^\star_{\boldsymbol{\theta}}}$ is $h_{\mu_{C^\star_{\boldsymbol{\theta}}}}(\boldsymbol{T_{a,b}}) \approx 2.315$.

Example 5.11. Set $\boldsymbol{a} = (0, 0.3, 0.7, 1)$, $\boldsymbol{b} = (0, 0.2, 0.4, 1)$ and consider the compatible transformation matrix $\boldsymbol{\theta}$, defined by

$$\boldsymbol{\theta} = \begin{pmatrix} 0 & 0.13 & 0.17 \\ 0.12 & 0 & 0.28 \\ 0.08 & 0.07 & 0.15 \end{pmatrix}.$$

Figure 6 contains the Lüroth-maps $T_{\boldsymbol{a}}$ and $T_{\boldsymbol{b}}$, respectively, Figure 7 depicts the densities of the first iterations of $\left(\mathcal{V}^{\boldsymbol{\theta}}_{\boldsymbol{a,b}}\right)^n(\Pi)$ for $n \in \{1, 2, 3, 5\}$.

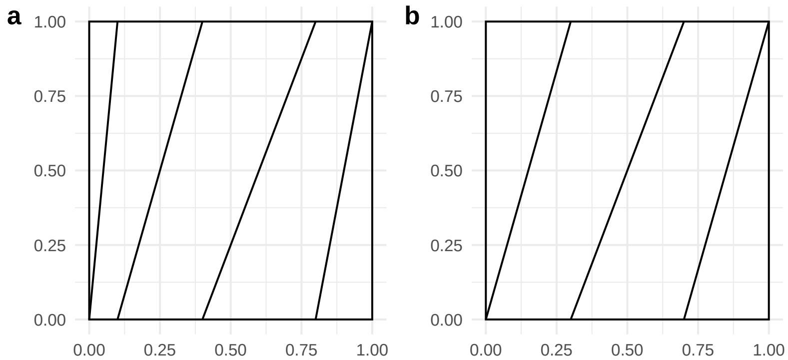

Figure 4: Lüroth-maps $T_{\boldsymbol{a}}$ (left panel - a) and $T_{\boldsymbol{b}}$ (right panel - b) according to Example 5.10.

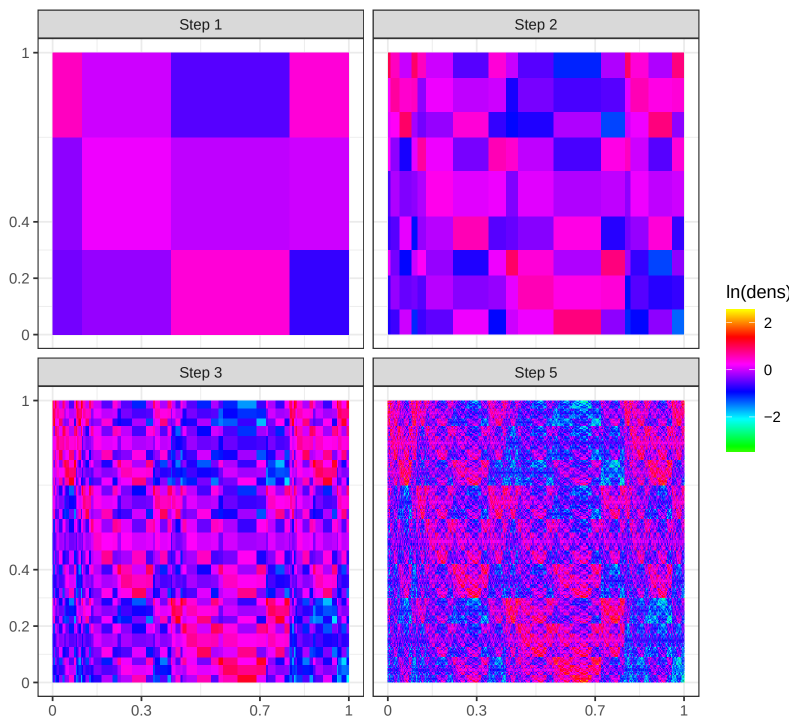

Figure 5: Image plot of the (natural) logarithm of the density of $\left(\mathcal{V}_{\boldsymbol{a},\boldsymbol{b}}^{\boldsymbol{\theta}}\right)^{(n)}(\Pi)$ for $n \in \{1, 2, 3, 5\}$ with $\boldsymbol{\theta}, \boldsymbol{a}$ and $\boldsymbol{b}$ according to Example 5.10.

According to Proposition 5.7 the invariant copula $C^\star_{\boldsymbol{\theta}}$ is singular and the support $K^\star_{\boldsymbol{\theta}}$ of $C^\star_{\boldsymbol{\theta}}$ fulfills $\lambda_2(K^\star_{\boldsymbol{\theta}}) = 0$. Moreover, the entropy of $\boldsymbol{T_{a,b}}$ w.r.t. to $\mu_{C^\star_{\boldsymbol{\theta}}}$ is $h_{\mu_{C^\star_{\boldsymbol{\theta}}}}(\boldsymbol{T_{a,b}}) \approx 1.85$.

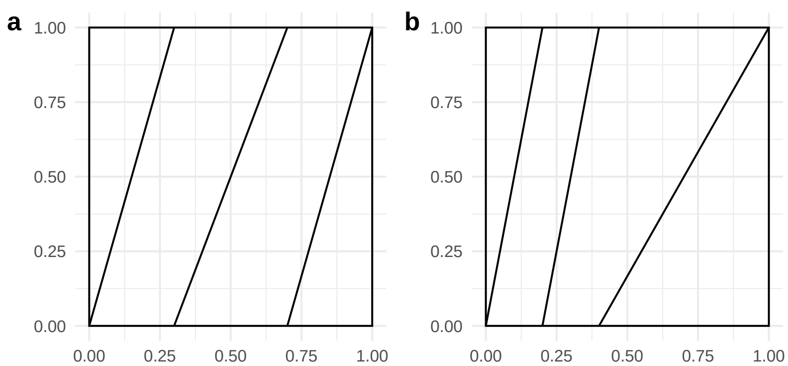

Figure 6: Lüroth-maps $T_{\boldsymbol{a}}$ (left panel - a) and $T_{\boldsymbol{b}}$ (right panel - b) according to Example 5.11.

Example 5.12. Set $\boldsymbol{a} = \left(0, \frac{1}{6}, \frac{5}{6}, 1\right)$, $\boldsymbol{b} = \left(0, \frac{1}{3}, \frac{2}{3}, 1\right)$ and consider the compatible transformation matrix $\boldsymbol{\theta}$ given by

$$\boldsymbol{\theta} = \begin{pmatrix} 0 & \frac{1}{6} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{6} & 0 \end{pmatrix}.$$

Figure 6 shows the Lüroth-maps $T_{\boldsymbol{a}}$ and $T_{\boldsymbol{b}}$, respectively, Figure 7 depicts the densities of the first iterations of $\left(\mathcal{V}^{\boldsymbol{\theta}}_{\boldsymbol{a,b}}\right)^n(\Pi)$ for $n \in \{1, 2, 3, 5\}$. According to Proposition 5.7 the invariant copula $C^\star_{\boldsymbol{\theta}}$ is singular and the support $K^\star_{\boldsymbol{\theta}}$ of $C^\star_{\boldsymbol{\theta}}$ fulfills $\lambda_2(K^\star_{\boldsymbol{\theta}}) = 0$. Moreover, the entropy of $\boldsymbol{T_{a,b}}$ w.r.t. to $\mu_{C^\star_{\boldsymbol{\theta}}}$ is $h_{\mu_{C^\star_{\boldsymbol{\theta}}}}(\boldsymbol{T_{a,b}}) \approx 1.33$.

For Lüroth double shuffles we can derive stronger versions of Theorem 4.4 and Theorem 4.6. As direct consequence of Theorem 3 in [24] the dynamical system $([0,1]^2, \mathcal{B}([0,1]^2), \mu_{C^\star_{\boldsymbol{\theta}}}, \boldsymbol{T_{a,b}})$ is isomorphic to a Bernoulli shift and as such strongly mixing. For every $A \in \mathcal{C}$ we will let $\mu_A = \mu_A^{\ll C^\star_{\boldsymbol{\theta}}} + \mu_A^{\perp C^\star_{\boldsymbol{\theta}}}$ denote the Lebesgue decomposition of μ_A w.r.t. $\mu_{C^\star_{\boldsymbol{\theta}}}$ and set $m_A^{\ll C^\star_{\boldsymbol{\theta}}} = \mu_A^{\ll C^\star_{\boldsymbol{\theta}}}([0,1]^2)$. The following theorem can be proved in completely the same manner as Theorem 4.4 - instead of the fact that $\boldsymbol{T_{a,b}}$ is strongly mixing w.r.t. $\lambda_2 = \mu_\Pi$ this time we use that $\boldsymbol{T_{a,b}}$ is strongly mixing w.r.t. $\mu_{C^\star_{\boldsymbol{\theta}}}$.

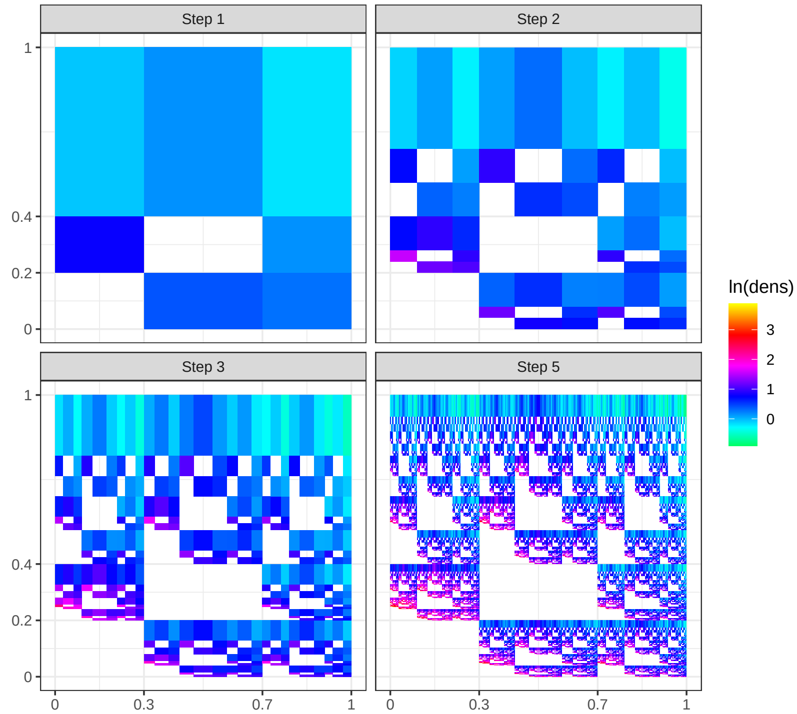

Figure 7: Image plot of the (natural) logarithm of the density of $\left(\mathcal{V}^{\boldsymbol{\theta}}_{\boldsymbol{a},\boldsymbol{b}}\right)^n(\Pi)$ for $n \in \{1, 2, 3, 5\}$ with $\boldsymbol{\theta}, \boldsymbol{a}$ and $\boldsymbol{b}$ according to Example 5.11.

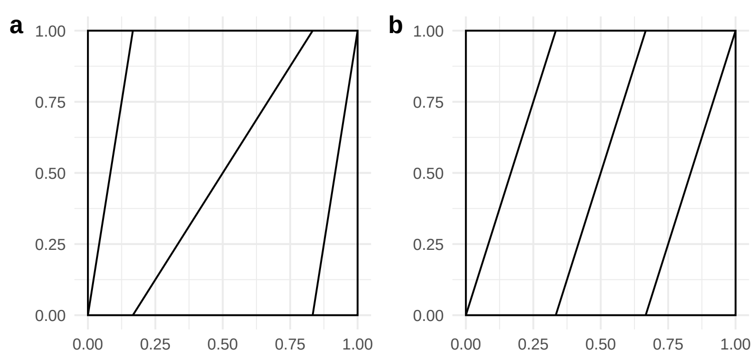

Figure 8: Lüroth-maps $T_{\boldsymbol{a}}$ (left panel - a) and $T_{\boldsymbol{b}}$ (right panel - b) according to Example 5.12.

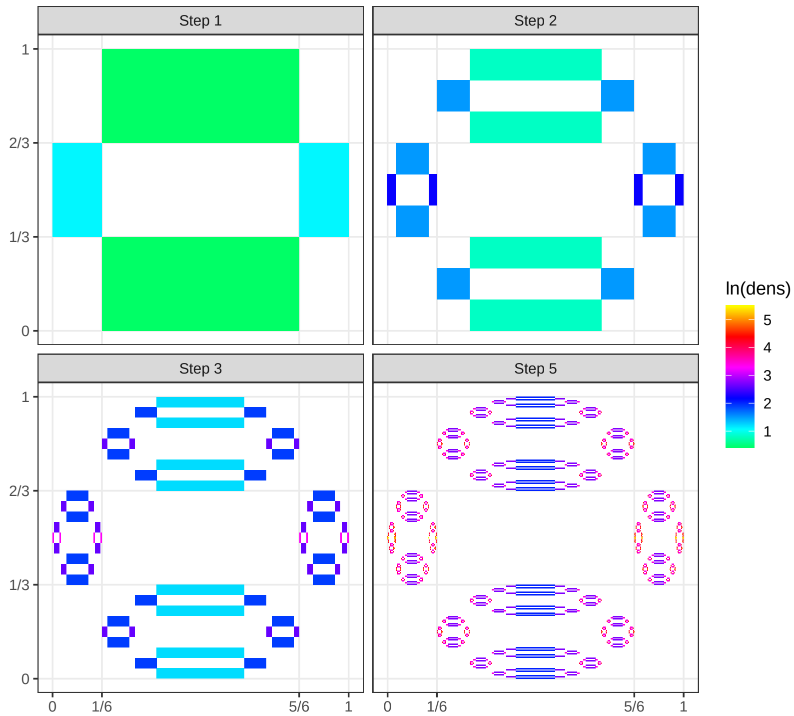

Figure 9: Image plot of the (natural) logarithm of the density of $\left(\mathcal{V}_{\boldsymbol{a},\boldsymbol{b}}^{\boldsymbol{\theta}}\right)^n(\Pi)$ for $n \in \{1, 2, 3, 5\}$ with $\boldsymbol{\theta}, \boldsymbol{a}$ and $\boldsymbol{b}$ according to Example 5.12.

Theorem 5.13. *Suppose that $A \in \mathcal{C}$ is $\mathcal{S}_{\boldsymbol{a},\boldsymbol{b}}$-invariant. If $m_A^{\ll C_{\boldsymbol{\theta}}^\star} = 1$ then $A = C_{\boldsymbol{\theta}}^\star$, if not then there exists a $\mathcal{S}_{\boldsymbol{a},\boldsymbol{b}}$-invariant copula $B \in \mathcal{C}$ fulfilling the following two properties:*

1. *B is singular with respect to $C_{\boldsymbol{\theta}}^\star$.*
2. *$A = m_A^{\ll C_{\boldsymbol{\theta}}^\star} \cdot C_{\boldsymbol{\theta}}^\star + (1 - m_A^{\ll C_{\boldsymbol{\theta}}^\star}) \cdot B$.*

Using the results from [24] as well as the above theorem yields the following stronger version of Theorem 4.6:

Theorem 5.14. *Suppose that $\boldsymbol{T}_{\boldsymbol{a},\boldsymbol{b}}$ is a bivariate Lüroth map and suppose that $\boldsymbol{\theta}, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2$ with $\boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_2$ are transformation matrices compatible with $\boldsymbol{T}_{\boldsymbol{a},\boldsymbol{b}}$. Then the following assertions hold:*

1. *$C_{\boldsymbol{\theta}_1}^\star$ and $C_{\boldsymbol{\theta}_2}^\star$ are singular with respect to each other.*

2. $C^\star_{\boldsymbol{\theta}}$ is an extreme point of $\Omega_{\boldsymbol{T_{a,b}}}$.

Proof. Since the first assertion is a direct consequence of Theorem 3 and Theorem 7 in [24] it suffices to prove the second one. Suppose there exist two copulas $A, B \in \Omega_{\boldsymbol{T_{a,b}}}$ and $\alpha \in (0,1)$ such that $C^\star_{\boldsymbol{\theta}} = \alpha A + (1-\alpha) D$ holds. Considering that A and B are absolutely continuous with respect to $C^\star_{\boldsymbol{\theta}}$ it follows that $m_A^{\ll C^\star_{\boldsymbol{\theta}}} = 1 = m_D^{\ll C^\star_{\boldsymbol{\theta}}}$, hence applying Theorem 5.13 yields $A = B = C^\star_{\boldsymbol{\theta}}$. In other words, $C^\star_{\boldsymbol{\theta}}$ is an extreme point of $\Omega_{\boldsymbol{T_{a,b}}}$. □

Remark 5.15. Notice that according to Theorem 5.14 each copula $C^\star_{\boldsymbol{\theta}}$ is an extreme point of $\Omega_{\boldsymbol{T_{a,b}}}$, it is however, only an extreme point of the whole family $\mathcal{C}$ if, and only if $\boldsymbol{\theta}$ (or its transpose $\boldsymbol{\theta}^t$) only contains rows having exactly one entry not equal to zero. In the latter case the resulting invariant copula $C^\star_{\boldsymbol{\theta}}$ (or its transpose) is completely dependent.

Considering Remark 5.3 yields the following nice corollary:

Corollary 5.16. *For each $\boldsymbol{T_{a,b}}$ the set $\Omega_{\boldsymbol{T_{a,b}}}$ contains uncountably many extreme points which are pairwise singular with respect to each other.*

Having in mind Choquet's famous theorem (see [19]) which implies that (in the current setting) for every $A \in \Omega_{\boldsymbol{T_{a,b}}}$ there exists some probability measure ϑ_A supported on the set of all extreme points of $\Omega_{\boldsymbol{T_{a,b}}}$ that 'represents' A and taking into account Remark 5.16 one might conjecture that we have already found ALL extreme points of $\Omega_{\boldsymbol{T_{a,b}}}$. The latter, however, is not true as the following examples show. To simplify notation we will write

$$\Lambda_{\boldsymbol{a},\boldsymbol{b}} := \{C^\star_{\boldsymbol{\theta}} \in \mathcal{C} \colon \boldsymbol{\theta} \text{ compatible with } \boldsymbol{T_{a,b}}\}.$$

Example 5.17. We show that in general $\Lambda_{\boldsymbol{a},\boldsymbol{b}}$ is not necessarily convex (compare with Example 3.5 and 3.6 in [11]) and consider $\boldsymbol{a} = \boldsymbol{b} = \big(0, \frac{1}{2}, 1\big)$. In this case the only two compatible transformation matrices containing a 0 are $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2$ given by

$$\boldsymbol{\theta}_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \boldsymbol{\theta}_2 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}.$$

In this case we obviously have $M = C^\star_{\boldsymbol{\theta}_1}$ as well as $W = C^\star_{\boldsymbol{\theta}_2}$. Figure 10 depicts the density plots of $\left(\mathcal{V}^{\boldsymbol{\theta}_1}_{\boldsymbol{a},\boldsymbol{b}}\right)^n(\Pi)$ and $\left(\mathcal{V}^{\boldsymbol{\theta}_2}_{\boldsymbol{a},\boldsymbol{b}}\right)^n(\Pi)$ for $n = 5$. Obviously $C := \frac{1}{2}W + \frac{1}{2}M$ is also $\mathcal{S}_{\boldsymbol{a},\boldsymbol{b}}$-invariant, however $C \notin \Lambda_{\boldsymbol{a},\boldsymbol{b}}$.

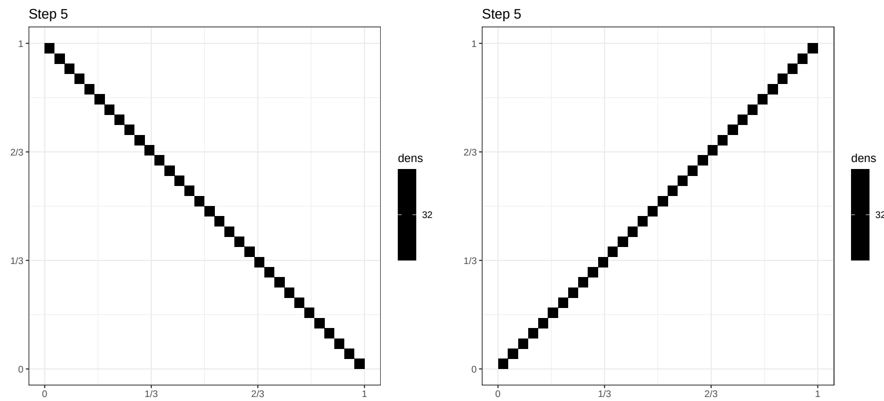

Figure 10: Image plot of the density of $\left(\mathcal{V}^{\boldsymbol{\theta}_1}_{\boldsymbol{a},\boldsymbol{b}}\right)^n(\Pi)$ for $n = 5$ (left panel) and of $\left(\mathcal{V}^{\boldsymbol{\theta}_2}_{\boldsymbol{a},\boldsymbol{b}}\right)^n(\Pi)$ (right panel), whereby $\boldsymbol{a} = \boldsymbol{b}$ are as in Example 5.17.

Example 5.18. We consider the same situation as in the previous example. Then, letting A_h denote the completely dependent copula with $h(x) = 2x(mod 1)$, according to Example 4.3 A_h is $\mathcal{S}_{\boldsymbol{a},\boldsymbol{b}}$-invariant. The copula A_h, however, is not a convex combination of M and W, hence it is not contained in the convex hull of the set $\Lambda_{\boldsymbol{a},\boldsymbol{b}}$.

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