ON SPECIAL PARTITIONS OF $[0,1]$ AND LINEABILITY WITHIN FAMILIES OF BOUNDED VARIATION FUNCTIONS

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Abstract. We show that there exists large algebraic structures (vector spaces, algebras, closed subspaces, etc.) formed entirely (except for 0), on one hand, by singular, nowhere monotonic functions on $[0,1]$ and, on the other hand, by absolutely continuous nowhere monotonic functions. Several tools, of independent interest, related to obtaining *special* partitions of $\mathbb{R}$ into uncountable collections will be provided and used. The results obtained in this note are either new or improved version of already existing ones.

1. Introduction and Preliminaries

This work is a contribution to the search for large vector spaces of functions having certain special (or pathological) property. Let us recall, for the sake of completeness, the following definitions of lineability and algebrability (that shall be recurrent throughout this text).

This terminology of lineable and spaceable coined by V.I. Gurariy and it was first introduced in [5, 39]. There has been plenty of work in this direction since its appearance about a decade ago. As a matter of fact, this notion was (just recently) introduced by the American Mathematical Society under the MSC2020 15A03 and 46B87 reference numbers. Let us, briefly, recall these notions.
Let α denote a cardinal number. A subset A of a vector space X is said to be

- *lineable* if $A \cup \{0\}$ contains an infinite dimensional vector space.
- *α-lineable* if $A \cup \{0\}$ contains an α-dimensional vector space.

If X is, in addition, a topological vector space, then A is called

- *spaceable* in X if $A \cup \{0\}$ contains a closed infinite dimensional vector subspace.

As introduced in [4], A is called *dense-lineable* in X if $A \cup \{0\}$ contains a dense vector subspace.
Of course, spaceability implies lineability and, if X is infinite-dimensional, then dense-lineability implies lineability too. Moreover, provided that X is a vector space contained in some (linear) algebra, then a set A is called:

- *algebrable* if there is an algebra M so that $M \setminus \{0\} \subset A$ and M is infinitely generated, that is, the cardinality of any system of generators of M is infinite.
- *strongly α-algebrable* if there exists an α-generated *free* algebra M with $M \setminus \{0\} \subset A$. Recall that if X is contained in a commutative algebra, then a set $B \subset X$ is a generating set of some free algebra contained in A if, and only if, for any $N \in \mathbb{N}$, any nonzero polynomial P in N variables without constant term and any distinct $f_1, \dots, f_N \in B$, we have $P(f_1, \dots, f_N) \neq 0$ and $P(f_1, \dots, f_N) \in A$.

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The notion of simple α-*algebrability* is defined in a similar fashion. Of course, strong α-algebrability implies α-algebrability, which implies α-lineability. However, in general, the converse implications do not hold, see, e.g., [3,8,10]. Recall that these notions of algebrability and their variants first appeared in [6,7]. The interested reader may also consult [2,12–16,18,20,22,23,31,33] for a complete account on lineability, spaceability, algebrability and related topics.

For measure spaces $(\Omega, \mathcal{A})$ and $(\Omega', \mathcal{A}')$, a measurable transformation $T : \Omega \to \Omega'$ and a measure μ on $(\Omega, \mathcal{A})$ we will let μ^T denote the push-forward of μ under T, i.e., $\mu^T(E') = \mu(T^{-1}(E'))$ for every $E' \in \mathcal{A}'$.

By $\Sigma_2 = \{0,1\}^{\mathbb{N}}$ we shall denote the code space of two symbols, and ρ will stand for the ultrametric defined by

$$\rho(\mathbf{k}, \mathbf{l}) = \begin{cases} 0 & \text{for } \mathbf{k} = \mathbf{l} \\ 2^{-\min\{i \in \mathbb{N}\colon k_i \neq l_i\}} & \text{for } \mathbf{k} \neq \mathbf{l}, \end{cases} \tag{1.1}$$

where $\mathbf{k} = (k_i)_{i \in \mathbb{N}}$, $\mathbf{l} = (l_i)_{i \in \mathbb{N}}$. The resulting metric space (Σ_2, ρ) is compact. Letting $\mathcal{B}(\Sigma_2)$ denote the Borel σ-field on Σ_2 and Σ_2' the subset of Σ_2 containing only elements $\mathbf{k} \in \Sigma_2$ without period 1 i.e., there is no i_0 such that $x_i = 1$ from i_0 on. We obviously have $\Sigma_2' \in \mathcal{B}(\Sigma_2)$.

Consider a function $f : [0,1] \to \mathbb{R}$. Recall that f is said to be of *bounded variation* whenever $TV(f;[0,1]) < \infty$, whereby the total variation of f on $[0,1]$ is defined by

$$TV(f;[0,1]) = \sup_{n \in \mathbb{N},\, 0 = x_0 \leq x_1 \leq x_2 \leq y_2 \leq \cdots \leq x_{n-1} \leq x_n = 1} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.$$

The function f is said to be *absolutely continuous* if it fulfills the following condition: given $\varepsilon > 0$, there is a $\delta > 0$ such that, for any $n \in \mathbb{N}$ and any collection of $2n$ points x_i, y_i satisfying $0 \leq x_1 \leq y_1 \leq x_2 \leq \cdots \leq x_n \leq y_n$ and $\sum_{i=1}^n (y_i - x_i) < \delta$, one has $\sum_{i=1}^n |f(y_i) - f(x_i)| < \varepsilon$. If f is absolutely continuous then it is continuous and of bounded variation, but the converse is false. Finally, f is called *singular* if it is continuous, bounded variation and $f'(x)$ exists and is zero almost everywhere on $[0,1]$. See [42] for background on this classes of functions.

This note is arranged as follows. Section 2 will focus on the construction of partitions of $\mathbb{R}$ into uncountable collections of so-called everywhere uncountably dense sets. Several different constructions shall be provided. These tools, although of interest by themselves, will be useful throughout this note. Section 3, on the other hand, shall deal with the family of all absolutely continuous nowhere monotonic functions, the family of singular nowhere monotonic functions, and the study of their lineability. New results will be obtained and known ones will be improved. We will write $\mathbb{N}_\infty = \{1,2,3,\ldots\} \cup \{\infty\}$ and refer to $\Sigma_\infty = (\mathbb{N}_\infty)^{\mathbb{N}}$ as code space of infinitely many symbols.

2. Everywhere dense partitions of $[0,1]$

In [44] (also see [26]) the authors construct partitions of the real line $\mathbb{R}$ into uncountable collections of so-called everywhere uncountably dense sets, whereby a partition $(E_t)_{t \in [0,1]}$ of $\mathbb{R}$ is called everywhere uncountably dense if, and only if, for every open set $\varnothing \neq U \subseteq \mathbb{R}$ and every $t \in [0,1]$ we have $\#(U \cap E_t) = \mathfrak{c} = \#\mathbb{R}$. There are various alternative simple ways to construct such partitions - we will describe one approach in detail and then only list some additional ones, all of them having in common that they are based on code spaces (also known as symbol dynamical systems) and impose conditions ignoring initial digits. To simplify notation we shall only work with $[0,1)$ (the extension to $\mathbb{R}$ is straightforward).

Let $\psi : [0,1) \to \Sigma_2'$ be the mapping assigning every $x \in [0,1)$ its unique binary expression $(x_1, x_2, \ldots) \in \Sigma_2'$ without period 1. Then obviously ψ is bijective and measurable and its inverse φ (extended to full Σ_2) is given by

$$\varphi(\mathbf{k}) = \sum_{i=1}^{\infty} \frac{k_i}{2^i}.$$

It is straightforward to verify that $\varphi : \Sigma_2 \to [0,1)$ is continuous, hence measurable, and injective on Σ_2'.

Let us denote by $\mathbf{1}_A$ the characteristic function of a set A. For every $n \in \mathbb{N}$ define the mapping $f_n : \Sigma_2 \to [0,1]$ by

$$f_n(\mathbf{k}) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{1\}}(k_i)$$

and set

$$h(\mathbf{k}) = \limsup_{n\to\infty} f_n(\mathbf{k}).$$

Considering that each f_n is obviously continuous, the induced mapping $h : \Sigma_2 \to [0,1]$ is measurable. Finally, for every $t \in [0,1]$ define the set $E_t \subseteq [0,1)$ by

$$E_t := (h \circ \psi)^{-1}(\{t\}) = \left\{ x \in [0,1) : \limsup_{n\to\infty} \frac{\#\{i \in \{1,\ldots,n\} : x_i = 1\}}{n} = t \right\}, \tag{2.1}$$

where $0.x_1x_2x_3\ldots$ is the unique binary representation of x without period 1.

Lemma 2.1. *The partition $(E_t)_{t\in[0,1]}$ according to equation (2.1) has the following properties:*

(1) $E_t \in \mathcal{B}([0,1))$ *for every* $t \in [0,1]$.
(2) $\lambda(E_t) = 0$ *for every* $t \neq \frac{1}{2}$, $\lambda(E_{\frac{1}{2}}) = 1$.
(3) $(E_t)_{t\in[0,1]}$ *is everywhere uncountably dense in* $[0,1)$.

Proof. The first assertion is trivial considering that $h \circ \psi$ is measurable, the fact that $\lambda(E_{\frac{1}{2}}) = 1$ is a direct consequence of the Strong Law of Large Numbers (see [30]) as well as the Birkhoff Ergodic Theorem (see [43]), using σ-additivity of λ therefore implies $\lambda(E_t) = 0$ for every $t \neq \frac{1}{2}$, and it remains to prove the third assertion.

We start with $t \in (0,1)$ and show that E_t is everywhere uncountably dense. For $\mathbf{k} \in \Sigma_2$ arbitrary but fixed let $\iota_t(\mathbf{k}) \in \Sigma_2$ be given by

$$\iota_t(\mathbf{k}) = (k_1, \underbrace{1,\ldots,1}_{n_1 \text{ times}}, k_2, \underbrace{0,\ldots,0}_{n_2 \text{ times}}, k_3, \underbrace{1,\ldots,1}_{n_3 \text{ times}}, k_4, \underbrace{0,\ldots,0}_{n_4 \text{ times}}, k_5, \ldots),$$

where $n_1, n_2, \ldots \in \mathbb{N}_0$ are as follows: n_1 is the minimal integer fulfilling $\frac{k_1+n_1}{n_1+1} \geq t$, n_2 the minimal integer fulfilling $\frac{k_1+k_2+n_1}{n_1+n_2+2} < t$, n_3 the minimal integer fulfilling $\frac{k_1+k_2+k_3+n_1+n_3}{n_1+n_2+n_3+3} \geq t$, and so on. Loosely speaking, sufficiently many 1s and 0s are inserted between $k_1, k_2, \ldots$ in such a way that the resulting sequence $(f_n(\iota_t(\mathbf{k})))_{n\in\mathbb{N}}$ jumps above and below t and converges to t.
It is straightforward to verify that $\iota_t : \Sigma_2 \to \Sigma_2$ is injective, hence considering $\varphi \circ \iota_t(\Sigma_2) \in E_t$ it follows that E_t has cardinality $\mathfrak{c}$. If $U \subseteq [0,1)$ is open and non-empty then U contains an interval of the form $[\frac{i-1}{2^n}, \frac{i}{2^n})$ for some $n \in \mathbb{N}$ and some $i \in \{1,\ldots,2^n\}$, so we can find some $l_1, \ldots, l_n \in \{0,1\}$ such that (slightly abusing notation) $\varphi((l_1, l_2, \ldots, l_n, \iota_t(\mathbf{k}))) \in E_t \cap U$ for every $\mathbf{k} \in \Sigma_2$, implying that $E_t \cap U$ has cardinality $\mathfrak{c}$. The cases $t = 0$ and $t = 1$ can be handled analogously. □

Alternative methods for constructing everywhere uncountably dense partitions $(E_t)_{t\in[0,1]}$ are the following ones:

(i) As usual, let $G : [0,1] \to [0,1]$ denote the Gauss map, defined by $G(0) = 0$ and $G(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$ for $x \in (0,1]$. Set $s_i = \frac{1}{i}$ for every $i \in \mathbb{N}$ and $s_\infty = 0$. Then the intervals $I_\infty = \{s_\infty\}, I_1 = (s_2, s_1], I_2 = (s_3, s_2], \ldots$ form a partition γ_G of $[0,1]$. Coding orbits of G via γ_G, the continued fraction expansion $\mathrm{cf} : [0,1] \to \Sigma_\infty$ is defined by setting $\mathrm{cf}(x) = \mathbf{a} = (a_1, a_2, a_3, \ldots) \in \Sigma$ if and only if $G^{i-1}(x) \in I_{a_i}$ holds for every $i \in \mathbb{N}$. It is well known that G is strongly mixing (hence ergodic) w.r.t. the absolutely continuous probability measure μ_G with density $\frac{1}{\ln 2}\frac{1}{1+x}$

for $x \in [0,1]$ (see [27]).

For every $t \in [0,1]$ define the set Ω_t by

$$\Omega_t := \left\{ \mathbf{k} \in \Sigma_\infty : \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\{1\}}(k_i) = t \right\} \tag{2.2}$$

and set $F_t = (\mathrm{cf})^{-1}(\Omega_t)$. Then it is straightforward to show that $(F_t)_{t\in[0,1]}$ is an everywhere uncountably dense partition of $[0,1]$ in Borel sets, and we have $\lambda(F_t) = 0$ for all $t \neq 2 - \frac{\log 3}{\log 2} =: t_1$ as well as $\lambda(F_{t_1}) = 1$.

(ii) The Gauss map G and the continued fraction expansion $\mathrm{cf} : [0,1] \to \Sigma_\infty$ allow to construct another uncountably dense partition of $[0,1]$: Suppose that $S \subseteq \mathbb{N}$ is a set containing at least two points. Then considering

$$\Omega_S := \left\{ \mathbf{k} \in \mathbb{N}^{\mathbb{N}} : \#\{i \in \mathbb{N} : k_i \notin S\} < \infty \text{ and } \#\{i \in \mathbb{N} : k_i = j\} = \aleph_0 \text{ for every } j \in S \right\}$$

and setting $H_S = (\mathrm{cf})^{-1}(\Omega_S)$ it is straightforward to verify that H_S fulfills $\#U \cap H_S = \mathfrak{c}$ for every non-empty open set $U \subseteq [0,1]$. Moreover, $\bigcup_{S\subset\mathbb{N},\#S\geq 2} H_S$ contains all $x \in [0,1]$ for which $\mathrm{cf}(x) \in \Sigma_\infty$ is not eventually constant. Letting Λ denote this very exceptional set, setting $H'_{\{1,2\}} = H_{\{1,2\}} \cup \Lambda$ as well as $H'_S = H_S$ for every S with $\#S \geq 2$ and $S \neq \{1,2\}$ therefore yields an everywhere uncountably dense partition of $[0,1]$.

(iii) Example 12.4 in [19] provides a less elementary example of an everywhere uncountably dense partition of $[0,1]$.

In the rest of this section we will construct countable partitions fulfilling a condition even stronger than uncountable nowhere denseness:

Definition 2.2. *A set $E \in \mathcal{B}([0,1])$ is called λ-dense if for every nonempty open subset $U \subseteq [0,1]$ we have $\lambda(E \cap U) > 0$ and $\lambda(E^c \cap U) > 0$. Moreover, a partition $(E_n)_{n\in\mathbb{N}}$ of $[0,1]$ is called λ-dense if every E_n is λ-dense.*

An example of a λ-dense set $E \in \mathcal{B}([0,1])$ fulfilling $\lambda(E) = \frac{1}{2}$, which was constructed via repeatedly copying shrunk versions of a fat Cantor set (a.k.a. Volterra-Smith-Cantor set) in the intervals constituting the complement of the initial Cantor set can be found in [21]. Moreover, it is known that the existence of λ-dense sets is equivalent to the existence of singular homeomorphisms, see [28]. In fact, let $h : [0,1] \to [0,1]$ denote an increasing singular homeomorphism (for methods of construction see, e.g., [17, 35, 38, 41]) and μ_h the probability measure associated to (the distribution function) h. Then there exists some $\Lambda \in \mathcal{B}([0,1])$ with $\lambda(\Lambda) = 1$ and $\mu_h(\Lambda) = 0$ such that $h'(x) = 0$ for every $x \in \Lambda$. Setting $g = \frac{1}{2}(h + id_{[0,1]})$ yields another increasing homeomorphism g and the set $g(\Lambda)$ is measurable and λ-dense: Measurability follows from the fact that $g' = \frac{1}{2}$ on Λ, hence $g(\Lambda)$ is a Borel set and $\lambda(g(\Lambda)) = \frac{1}{2}$ holds (see [34]). Letting μ_g denote the probability measure corresponding to (the distribution function) g we have $\mu_g = \frac{1}{2}(\mu_h + \lambda)$, hence using $\lambda^{g^{-1}} = \mu_g$ and letting $(a,b) \subseteq [0,1]$ be non-empty yields

$$\begin{aligned} \lambda\left(g(\Lambda) \cap (a,b)\right) &= \lambda\left(g(\Lambda) \cap g \circ g^{-1}(a,b)\right) = \lambda^{g^{-1}}\left(\Lambda \cap g^{-1}(a,b)\right) \\ &= \mu_g\left(\Lambda \cap g^{-1}(a,b)\right) = \tfrac{1}{2}\lambda\left(\Lambda \cap g^{-1}(a,b)\right) \\ &= \tfrac{1}{2}\lambda\left((g^{-1}(a), g^{-1}(b))\right) > 0. \end{aligned}$$

To construct λ-dense partitions of $[0,1]$ the subsequent simple lemma will prove useful. We start with some notation and then state the main properties of the transformation which will be used in the sequel.

Fix $a \in (0,1]$ and suppose that $E \in \mathcal{B}([0,1])$ fulfills $\lambda(E) = a$. Set $\vartheta_E = \lambda|_E$ and define the function $T : [0,1] \to [0,a]$ by

$$T(x) = \int_{[0,x]} \mathbf{1}_E \, d\lambda = \lambda(E \cap [0,x]) = \vartheta_E([0,x]). \tag{2.3}$$

The following lemma gathers some properties of T, implying that T is an isomorphism (see [43]) of $(E, E \cap \mathcal{B}([0,1]), \vartheta_E)$ and $([0,a], \mathcal{B}([0,a]), \lambda|_{[0,a]})$.

Lemma 2.3. *The function T defined by equation* (2.3) *enjoys the following properties:*

(1) *T is absolutely continuous, non-decreasing and fulfills $T(0) = 0, T(1) = a$.*
(2) *There exists a set $\widehat{E} \in \mathcal{B}([0,1])$ fulfilling $\widehat{E} \subseteq E$ and $\lambda(\widehat{E}) = \lambda(I)$ such that $T|_{\widehat{E}}$ is injective.*
(3) $\vartheta_E^T = \lambda|_{[0,a]}$.
(4) $\lambda(T(E)) = \lambda(T(\widehat{E})) = a$.
(5) *Defining $S : T(\widehat{E}) \to \widehat{E}$ by $S(T(x)) = x$ we have $\lambda^S_{|[0,a]} = \vartheta_E$.*

Proof. The first assertion is obvious. Setting $\Lambda := \{x \in (0,1) : T'(x) = \mathbf{1}_E(x)\}$, according to [37] we have $\Lambda \in \mathcal{B}([0,1])$ as well as $\lambda(\Lambda) = 1$. Considering $\widehat{E} := E \cap \Lambda$ we get $\widehat{E} \in \mathcal{B}([0,1])$ and $\lambda(\widehat{E}) = a$. If $x, y \in \widehat{E}$ and $x < y$, then $T'(x) = 1$, from which $T(y) \geq T(x + \frac{1}{n}) > T(x) + \frac{1}{2n}$ follows for all n greater than some index $n_0 \in \mathbb{N}$. This proves (2).

To prove the third assertion interpret $X = id_{[0,1]}$ as random variable on the probability space

$$([0,1], \mathcal{B}([0,1]), \frac{1}{a}\,\vartheta_E).$$

Then the distribution function F_X of X coincides with $\frac{1}{a}\,T$, so, by the probability integral transform, the random variable $Y = \frac{1}{a}\,T \circ X$ fulfills $Y \sim \mathcal{U}(0,1)$ (that is, Y has uniform distribution on $[0,1]$), from which $\vartheta_E^T([0,b]) = b$ for every $b \in [0,a]$ follows immediately. Considering $T([0,1]) = [0,a]$ this implies $\vartheta_E^T = \lambda|_{[0,a]}$. Since T is injective on $\widehat{E}$ according to [40, Theorem 4.5.4] we have $T(\widehat{E}) \in \mathcal{B}([0,1])$, hence

$$a = \lambda([0,a]) \geq \lambda(T(\widehat{E})) = \vartheta_E^T(T(\widehat{E})) \geq \vartheta_E(\widehat{E}) = a$$

shows assertion (4) (notice that $\lambda(T(E))$ is well-defined since $[0,a] \setminus T(\widehat{E}) \in \mathcal{B}([0,1])$ has λ-measure 0, implying that $T(E)$ is a Lebesgue-measurable set). The fifth assertion can be proved analogously. □

In case $E \in \mathcal{B}([0,1])$ is λ-dense the following stronger version of assertion (2) in Lemma 2.3 holds:

Corollary 2.4. *Suppose that $E \in \mathcal{B}([0,1])$ fulfills $\lambda(E) = a \in (0,1]$ and that E is λ-dense in $[0,1]$. Then the function T defined according to equation* (2.3) *is injective on $[0,1]$ and we have $\lambda|_{[0,a]}^{T^{-1}} = \vartheta_E$. Moreover $|T(x) - T(y)| < |x - y|$ holds for all $x, y \in [0,1]$ with $x \neq y$.*

Proof. Injectivity of T and $\lambda|_{[0,a]}^{T^{-1}} = \vartheta_E$ are a direct consequence of Lemma 2.3 and λ-densensess of E. The third assertion follows directly from the fact that for $x < y$ we have

$$T(y) - T(x) = \int_{(x,y)} \mathbf{1}_E \, d\lambda = \lambda(E \cap (x,y)) \in (0, y - x)$$

whereby the last inequality is strict since $\lambda(E^c \cap (x,y)) > 0$. □

Using the afore-mentioned results a λ-dense partition $(E_n)_{n \in \mathbb{N}}$ can be constructed as follows:
(i) Suppose that $E_1 \in \mathcal{B}([0,1])$ is λ-dense with $\lambda(E_1) = \frac{1}{2}$ and fulfills $\{0,1\} \subseteq E_1$. Set $E := E_1 \setminus \{0,1\}$ and $E \cdot \frac{1}{2} := \{\frac{x}{2} : x \in E\}$. Defining $T_1 : [0,1] \to [0, \frac{1}{2}]$ and E_2 by

$$T_1(x) = \int_{[0,x]} \mathbf{1}_{E_1^c} \, d\lambda = \lambda(E_1^c \cap [0,x]) = \vartheta_{E_1^c}([0,x]), \quad E_2 = T_1^{-1}(E \cdot \tfrac{1}{2}) \cap E_1^c$$

then using Lemma 2.3 and Corollary 2.4 it follows that for every nonempty open set $U \subseteq [0,1]$ we have

$$\begin{aligned} \lambda(E_2 \cap U) \quad &\geq \quad \vartheta_{E_1^c}(E_2 \cap U) = \vartheta_{E_1^c}(T_1^{-1}(E \cdot \tfrac{1}{2}) \cap T_1^{-1} \circ T_1(U)) = \vartheta_{E_1^c}^{T_1}((E \cdot \tfrac{1}{2}) \cap T_1(U)) \\ &= \quad \lambda((E \cdot \tfrac{1}{2}) \cap T_1(U)) > 0, \end{aligned}$$

whereby the last inequality follows from the fact that $T_1(U)$ is open since T_1 is a homeomorphism of $[0,1]$ and $[0,\frac{1}{2}]$ and $E\cdot\frac{1}{2}$ is λ-dense in $[0,\frac{1}{2}]$. Considering $E_2^c \supseteq E_1$ we obviously also have $\lambda(E_2^c\cap U) > 0$, so E_2 is λ-dense. Also notice that $\lambda(E_2) = \frac{1}{4}$ since

$$\lambda(E_2) = \vartheta_{E_1^c}^{T_1}\left(E_1 \cdot \tfrac{1}{2}\right) = \lambda_{[0,\frac{1}{2}]}(E\cdot\tfrac{1}{2}) = \tfrac{1}{4}$$

holds. The set $E_1 \cup E_2$ is λ-dense too since for every non-empty open U on the one hand we have $\lambda((E_1\cup E_2)\cap U) \geq \lambda(E_1 \cap U) > 0$, and on the other hand

$$\begin{aligned}\lambda\left((E_1\cup E_2)^c \cap U\right) &\geq \vartheta_{E_1^c}(E_2^c\cap U) = \vartheta_{E_1^c}\left(T_1^{-1}(E\cdot\tfrac{1}{2})^c\cap U\right) = \vartheta_{E_1^c}\left(T_1^{-1}(E\cdot\tfrac{1}{2})^c \cap T_1^{-1}\circ T_1(U)\right)\\ &= \vartheta_{E_1^c}^{T_1}\left((E\cdot\tfrac{1}{2})^c\cap T_1(U)\right) = \lambda\left((E\cdot\tfrac{1}{2})^c \cap T_1(U)\right) > 0.\end{aligned}$$

(ii) Define the transformation $T_2 : [0,1] \to [0,\frac{1}{4}]$ and the set E_3 by

$$T_2(x) = \int_{[0,x]} \mathbf{1}_{(E_1\cup E_2)^c}\, d\lambda = \lambda(E_1^c \cap E_2^c \cap [0,x]) = \vartheta_{(E_1\cup E_2)^c}([0,x]), \quad E_3 = T_2^{-1}(E\cdot\tfrac{1}{4}) \cap E_1^c \cap E_2^c.$$

Using the same arguments as before we get that E_3 and $\bigcup_{i=1}^3 E_i$ are λ-dense, and that $\lambda(E_3) = \frac{1}{8}$.
(iii) Proceeding inductively we finally get a sequence $(E_n)_{n\in\mathbb{N}}$ of pairwise disjoint, λ-dense Borel sets fulfilling $\lambda(\bigcup_{i=1}^\infty E_i) = \sum_{i=1}^\infty \lambda(E_i) = 1$. In case of $\bigcup_{i=1}^\infty E_i = [0,1]$ the construction is complete, in case of $\bigcup_{i=1}^\infty E_i \neq [0,1]$ defining $E_1' := E_1 \cup ([0,1]\setminus \bigcup_{i=1}^\infty E_i)$ as well as $E_i' = E_i$ for every $i \geq 2$ yields a λ-dense partition $(E_n')_{n\in\mathbb{N}}$.

Recall that a positive measure μ defined on the Borel σ-algebra of a topological space is said to have *full support* whenever $\mu(U) > 0$ for every nonempty open set U. Extending Definition 2.2 in the natural way to general probability measures on $\mathcal{B}([0,1])$ the following result holds:

Theorem 2.5. *Suppose that μ is a probability measure on $\mathcal{B}([0,1])$ with full support. Then there exists a μ-dense partition $(D_n)_{n\in\mathbb{N}}$ of $[0,1]$.*

Proof. Suppose that $(E_n)_{n\in\mathbb{N}}$ is a λ-dense partition of $[0,1]$ and let $F = F_\mu$ denote the distribution function corresponding to μ. Then $F : [0,1] \to [0,1]$ is a homeomorphism and each set $D_n := F^{-1}(E_n), n \in \mathbb{N}$, is measurable. If $U \subseteq [0,1]$ is non-empty and open then, using the fact that obviously $\mu^F = \lambda$, we get

$$\mu(D_n \cap U) = \mu(F^{-1}(E_n)\cap F^{-1}\circ \underbrace{F(U)}_{\text{open}}) = \mu^F(E_n \cap F(U)) = \lambda(E_n \cap F(U)) > 0,$$

the fact that $\mu(D_n^c \cap U) > 0$ follows in the same manner. Since $(D_n)_{n\in\mathbb{N}}$ is a partition of $[0,1]$, this completes the proof. ☐

3. Large classes of nowhere monotonic functions with bounded variation on $[0,1]$

In what follows BV will denote the family of all real-valued functions f with bounded variation on $[0,1]$. To simplify notation we will also write $TV(f)$ instead of $TV(f;[0,1])$. Defining the total variation norm $\|f\|_{BV}$ of $f \in BV$ by

$$\|f\|_{BV} = |f(0)| + TV(f) \tag{3.1}$$

it is well known that $(BV, \|\cdot\|_{BV})$ is a (non-separable) Banach space (see, e.g., [1]), that a function is of bounded variation if, and only if, it can be expressed as the difference of two monotonic functions (see [11]), and that for absolutely continuous f the total variation can be calculated as (again see [11])

$$TV(f) = \int_{[0,1]} |f'|\, d\lambda, \tag{3.2}$$

whereby we (here and subsequently) set $f'(x) = 0$ for every x from the set (of λ-measure 0) of all $x \in [0,1]$ at which f is not differentiable. Recall also that f is absolutely continuous if and only if there is a λ-integrable function g on $[0,1]$ such that Barrow's rule $\int_{[0,x]} g\,d\lambda = f(x) - f(0)$ holds for all $x \in [0,1]$, in which case $f' = g$ λ-almost everywhere in $[0,1]$ (see [42, Chapter 6]).

Suppose now that $(E_n)_{n\in\mathbb{N}}$ is a λ-dense partition of $[0,1]$. Based on $(E_n)_{n\in\mathbb{N}}$ we are going to construct very large subsets of $(BV, \|\cdot\|_{BV})$ consisting exclusively of functions that are nowhere monotonic (i.e., not monotonic on any non-degenerated interval $(a,b) \subseteq [0,1]$). For every $n \in \mathbb{N}$ define the absolutely continuous function $d_n \in BV$ by

$$d_n(x) = \frac{2^{2n}}{3} \int_{[0,x]} \left(\mathbf{1}_{E_{2n-1}} - \mathbf{1}_{E_{2n}}\right) d\lambda. \tag{3.3}$$

A straightforward calculation using equation (3.2) shows that $\|d_n\|_{BV} = TV(d_n) = 1$ holds for every $n \in \mathbb{N}$. Using $(d_n)_{n\in\mathbb{N}}$ we can now construct an embedding κ of $(l_1, \|\cdot\|_1)$ into $(BV, \|\cdot\|_{BV})$ by

$$\kappa(\mathbf{a}) = \sum_{n=1}^{\infty} a_n d_n, \tag{3.4}$$

whereby $\mathbf{a} = (a_1, a_2, \ldots)$ denotes an arbitrary element of l_1. The subsequent lemma summarizes key properties of κ.

Lemma 3.1. *The operator κ defined according to equation* (3.4) *is an isometric embedding of the Banach space $(l_1, \|\cdot\|_1)$ in $(BV, \|\cdot\|_{BV})$. Moreover, $\kappa(l_1)$ is a closed, infinite-dimensional subspace of $(BV, \|\cdot\|_{BV})$.*

Proof. For every $n \in \mathbb{N}$ there exists a Borel set Λ_n with $\lambda(\Lambda_n) = 1$ such that

$$d_n'(x) = \frac{2^{2n}}{3}(\mathbf{1}_{E_{2n-1}}(x) - \mathbf{1}_{E_{2n}}(x))$$

holds for every $x \in \Lambda_n$. Consequently, the set $\Lambda = \bigcap_{n=1}^{\infty} \Lambda_n \in \mathcal{B}([0,1])$ fulfills $\lambda(\Lambda) = 1$ too. For every $x \in \Lambda$ and every $\mathbf{a} \in l_1$ we have

$$\left|\sum_{i=1}^{\infty} a_i d_i'(x)\right| = \sum_{i=1}^{\infty} |a_i d_i'(x)| = \sum_{i=1}^{\infty} |a_i| \cdot \frac{2^{2n}}{3} \cdot \mathbf{1}_{E_{2i-1}\cup E_{2i}}(x).$$

Applying the Monotone Convergence Theorem (see [37]) yields

$$\int_{[0,1]} \left|\sum_{i=1}^{\infty} a_i d_i'(x)\right| d\lambda(x) = \int_{[0,1]} \sum_{i=1}^{\infty} |a_i|\, \tfrac{2^{2i}}{3}\, \mathbf{1}_{E_{2i-1}\cup E_{2i}}(x)\, d\lambda(x) = \sum_{i=1}^{\infty} |a_i| = \|\mathbf{a}\|_1,$$

hence the function $x \mapsto \int_{[0,x]} \sum_{i=1}^{\infty} a_i d_i'\, d\lambda$ is integrable. Using Dominated Convergence and Fubini's Theorem we get

$$\begin{aligned} \int_{[0,x]} \sum_{i=1}^{\infty} a_i d_i'(t)\, d\lambda(t) &= \lim_{n\to\infty} \int_{[0,x]} \sum_{i=1}^{n} a_i d_i'(t)\, d\lambda(t) = \lim_{n\to\infty} \sum_{i=1}^{n} a_i d_i(x) \\ &= \kappa(\mathbf{a})(x) = \kappa(\mathbf{a})(x) - \kappa(\mathbf{a})(0), \end{aligned}$$

so $\kappa(\mathbf{a})$ is absolutely continuous with density $\sum_{i=1}^{\infty} a_i d_i'(x)$, and applying equation (3.2) yields

$$\|\kappa(\mathbf{a})\|_{BV} = \|\mathbf{a}\|_1,$$

which completes the proof of the first assertion of the theorem. The second assertion now follows immediately since $\kappa(l_1)$ obviously is a subspace of $(BV, \|\cdot\|_{BV})$ and it is closed since κ is an isometry. In fact, letting $(\kappa(\mathbf{a}_n))_{n\in\mathbb{N}}$ denote a convergent sequence in $(BV, \|\cdot\|_{BV})$ with a limit $f \in BV$, then from the fact that κ is an isometry it follows that $(\mathbf{a}_n)_{n\in\mathbb{N}}$ is a convergent sequence in $(l_1, \|\cdot\|_1)$ which has some limit $\mathbf{a} \in l_1$, and we have $\kappa(\mathbf{a}) = f$. □

Theorem 3.2. *The family of all absolutely continuous, nowhere monotonic functions is spaceable in $(BV, \|\cdot\|_{BV})$.*

Proof. Considering Lemma 3.1 it only remains to show that each $\kappa(\mathbf{a})$ with $\mathbf{a} \neq \mathbf{0}$ is nowhere monotonic on $[0,1]$. Let $(c,d) \subseteq [0,1]$ with $c < d$ be arbitrary but fixed. Letting n_0 denote the smallest integer such that $a_n \neq 0$ it follows that the sets Λ_+, Λ_-, defined by

$$\Lambda_+ = \left\{t \in (c,d) : (\kappa(\mathbf{a}))'(t) = a_{n_0} \tfrac{2^{2n_0}}{3}\right\}, \quad \Lambda_- = \left\{t \in (c,d) : (\kappa(\mathbf{a}))'(t) = -a_{n_0} \tfrac{2^{2n_0}}{3}\right\},$$

fulfill $\lambda(\Lambda_+), \lambda(\Lambda_-) > 0$, implying that $\kappa(\mathbf{a})$ cannot be monotonic on (c,d). □

Remark 3.3. *An important result that is related to Theorem 3.2 is the following negative one due to Gurariy* [25]*: The family of all differentiable functions in* $[0,1]$ *is not spaceable in the space of continuous functions under the supremum norm.*

Next we are interested in studying the dense lineability of the family of absolutely continuous, nowhere monotonic functions. However, in order to achieve this, we need to recall some tools that will be very useful to accomplish this result (see [4]).

Definition 3.4. *Let A, B be subsets of a vector space X. We say that A is stronger than B if $A + B \subseteq A$.*

Theorem 3.5. *Let X be a separable Banach space, and consider two subsets A, B of X such that A is lineable and B is dense-lineable. If A is stronger than B, then A is dense-lineable.*

Now we are ready to state and prove the result we announced earlier. By $(\mathcal{C}[0,1], \|.\|_\infty)$ we shall denote, as usual, the Banach space of all continuous functions $[0,1] \to \mathbb{R}$, endowed with the supremum norm. Recall that this space is separable.

Theorem 3.6. *The family of all absolutely continuous, nowhere monotonic functions is dense lineable in $(\mathcal{C}[0,1], \|\cdot\|_\infty)$.*

Proof. Let $\{E_n\}_{n\in\mathbb{N}}$ be a λ-dense partition with

$$\lambda(E_n) = 1/2^n \text{ for all } n \in \mathbb{N}.$$

Now, let us consider a bijection $\varphi : \mathbb{N} \times \mathbb{N} \to 2\mathbb{N}$. Given $n \in \mathbb{N}$, let us define the function $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) := \begin{cases} \dfrac{2^{\varphi(n,m)-1}}{(\varphi(n,m)-1)^2} & \text{if } x \in E_{\varphi(n,m)-1}, \\ -\dfrac{2^{\varphi(n,m)}}{\varphi(n,m)^2} & \text{if } x \in E_{\varphi(n,m)}. \end{cases}$$

It is easy to see that f_n is Lebesgue-integrable. Now, we set $F_n(x) := \int_{[0,x]} f_n \, d\lambda$ for every $x \in [0,1]$.

Plainly, every F_n is absolutely continuous, and there is a measurable set Z_n with $\lambda(Z_n) = 0$ such that $F_n'(x) = f_n(x)$ for all $x \in [0,1] \setminus Z_n$. We set $Z := \bigcup_{n\in\mathbb{N}} Z_n$, so that $\lambda(Z) = 0$.

Now, we denote by A_0 the family of all absolutely continuous, nowhere monotonic functions. In order to show that A_0 is dense lineable, it is enough to prove that A is, where A is defined as the set of all absolutely continuous functions $f : [0,1] \to \mathbb{R}$ satisfying the following property:

(U) If we set an interval $(a,b) \subset [0,1]$ and $M > 0$, there exist two subsets of (a,b), of positive measure, in such a way that f is differentiable on both of them and having derivative bigger than M on the first set and smaller than $-M$ on the second one.

It is clear that $A \subset A_0$. The functions F_n's are linearly independent (since the f_n's have pairwise disjoint supports). Next, fix $F \in \operatorname{span}\{F_n\}_{n\geq 1} \setminus \{0\}$, so that there are $p \in \mathbb{N}$ and reals $c_1, \ldots, c_p$ with $c_p \neq 0$ such that $F = c_1 F_1 + \cdots + c_p F_p$. Then F is absolutely continuous. Let us prove that $F \in A$. Since A is invariant under scaling, we can suppose that $c_p = 1$. We have to show that F satisfies (U).

To this end, fix $M > 0$ as well as an interval $(a, b) \subset [0, 1]$. Since $\lim_{m\to\infty} \varphi(p, m) = +\infty = \lim_{k\to\infty} \frac{2^k}{k}$, there is $m \in \mathbb{N}$ such that

$$\begin{cases} F_p'(x) = f_p(x) = \dfrac{2^{\varphi(p,m)-1}}{(\varphi(n,m)-1)^2} > M & \text{for all } x \in E_{\varphi(p,m)-1} \setminus Z, \text{ and} \\ F_p'(x) = f_p(x) = -\dfrac{2^{\varphi(p,m)}}{\varphi(n,m)^2} < -M & \text{for all } x \in E_{\varphi(p,m)} \setminus Z. \end{cases}$$

Observe that, since the sets E_n's are pairwise disjoint, we have $F_j'(x) = f_j(x) = 0$ for all $j \in \{1, \ldots, p-1\}$ (the last set is meant as $\varnothing$ if $p = 1$) and all $x \in C_1 \cup C_2$, where we have set

$$C_1 := (E_{\varphi(p,m)-1} \setminus Z) \cap (a, b) \ \text{ and } \ C_2 := (E_{\varphi(p,m)} \setminus Z) \cap (a, b).$$

Since the partition $\{E_n\}_{n\geq 1}$ is λ-dense, we get $\lambda(C_1) > 0 < \lambda(C_2)$. Therefore $F'(x) = f_p(x) + \sum_{j=1}^{\infty} c_j f_j(x) = f_p(x) > M$ for all $x \in C_1$ and, analogously, $F'(x) < -M$ for all $x \in C_2$. Hence F satisfy (U), and so $\operatorname{span}\{F_n\}_{n\geq 1} \subset A \cup \{0\}$, which yields that A is lineable.

Finally, let us consider B to be the set of polynomials, which is dense in $X := (\mathcal{C}[0, 1], \|\cdot\|_\infty)$ by the Weierstrass approximation theorem. But B is a vector space in itself; therefore it is dense lineable in X. Let us check that $A + B \subseteq A$. Let P be a polynomial and $f \in A$. It is clear that $P + f$ is absolutely continuous. Clearly, P' is bounded in $[0, 1]$. Let us take any bound, γ, of $|P'|$ in $[0, 1]$. Since f enjoys property (U), if we take $M > 0$ and any interval $(a, b) \subset [0, 1]$, we can find two subsets $S_1, S_2 \subset (a, b)$, of positive measure, in which f is differentiable and $(-1)^i f'(x) > \gamma + M$ for $x \in S_i$ with $i \in \{1, 2\}$. Therefore, at the points of S_1 we have that $P' + f'$ is bigger than M and, at the points in S_2 is less than $-M$. That is, $P + f \in A$, which yields $A + B \subseteq A$. Thus, we can apply Theorem 3.5, and we obtain that A is dense lineable in $(\mathcal{C}[0, 1], \|\cdot\|_\infty)$, as required. □

Remarks 3.7. *1. In the preceding theorem, the space $\mathcal{C}[0, 1]$ cannot be replaced with BV, because the set of absolutely continuous functions is not dense in the latter space. In fact, it is closed in BV and separable under the inherited topology, while BV is not separable (see* [1]*).*
2. In the proof of Theorem 3.6, one is tempted to directly choose the family of all absolutely continuous, nowhere monotonic functions as the set A in order to apply Theorem 3.5. But in this case $A + B$ is not contained in A. Indeed, in 1974 Katznelson and Stromberg [29] *provided a nowhere monotonic, everywhere differentiable function $L : \mathbb{R} \to \mathbb{R}$ with $|L'(x)| \leq 1$ for all $x \in \mathbb{R}$. Then its restriction to $[0, 1]$ is nowhere monotonic and absolutely continuous. Now, the function $P(x) := 2x$ is in B, but $L + P \notin A$ as it is strictly increasing, because $(L + P)' = 2 + L' > 0$ on $[0, 1]$.*

We now focus on studying the algebrability of the family of all absolutely continuous, nowhere monotonic functions on $[0, 1]$. For proving the main result the following lemma, which is a slight modification of a result from set theory (see [36]) will be key:

Lemma 3.8. *There exists a family $\mathcal{D} = (D_t)_{t\in\mathbb{R}}$ of pairwise different subsets of $\mathbb{N}$ having the following three properties:*

(P1) *For all $n, m \in \mathbb{N}$ with $n \leq m$ and pairwise different real numbers $t_1, t_2, \ldots, t_n, t_{n+1}, \ldots, t_{n+m}$ we have*

$$D_{t_1} \cap \ldots \cap D_{t_n} \cap D_{t_{n+1}}^c \cap \ldots \cap D_{t_m}^c \neq \varnothing. \tag{3.5}$$

(P2) *The intersections in (3.5) have cardinality $\aleph_0$.*
(P3) *For every $i \in \mathbb{N}$ we have $D_i \cap \{1, 2, \ldots, i\} = \{i\}$.*

Theorem 3.9. *The family of all absolutely continuous, nowhere monotonic functions is strongly $\mathfrak{c}$-algebrable.*

Proof. Let $\mathcal{D}$ be a family fulfilling the properties from Lemma 3.8, $(d_n)_{n\in\mathbb{N}}$ the family of functions defined according to equation (3.3), and suppose that $\mathbf{a} \in l_1$ contains no 0. For every $D \in \mathcal{D}$ define the function $f_D : [0,1] \to \mathbb{R}$ by

$$f_D(x) = \sum_{n=1}^{\infty} \mathbf{1}_D(n) a_n d_n.$$

Then proceeding as in the proof of Lemma 3.1 shows that f_D is absolutely continuous with derivative $f'_D(x) = \sum_{n=1}^{\infty} \mathbf{1}_D(n) a_n d'_n(x)$ for λ-almost every $x \in [0,1]$, and that f is nowhere monotonic. We are going to show that $\{f_D : D \in \mathcal{D}\}$ is algebraically independent and that it generates an algebra contained in the family of absolutely continuous, nowhere monotonic functions. Suppose that $m \in \mathbb{N}$ is arbitrary but fixed, that $p : \mathbb{R}^m \to \mathbb{R}$ is a non-zero polynomial of degree r without constant term and that $D_{t_1}, \ldots, f_{D_{t_m}}$ are different elements in $\mathcal{D}$. To simplify notation we will write

$$F_p = p(f_{D_{t_1}}, \ldots, f_{D_{t_m}}).$$

We shall proceed by induction of the degree r of p and to prove that F_p is absolutely continuous and nowhere monotonic.

(i) Suppose that $r = 1$. Then F_p is of the form $F_p = \sum_{j=1}^{m} \beta_j f_{D_{t_j}}$ for some constants $\beta_1, \ldots, \beta_m$, not all of them being 0, and we have

$$F'_p(x) = \sum_{j=1}^{m} \beta_j \sum_{n=1}^{\infty} \mathbf{1}_{D_{t_j}}(n) a_n d'_n(x)$$

for λ-almost every $x \in [0,1]$. Set $j_0 := \min\{j \in \{1,\ldots,m\} : \beta_j \neq 0\}$ and suppose that $n_0 \in D_{t_{j_0}} \cap \bigcap_{j\neq j_0}^{m} D^c_{t_j}$. Then for λ-almost every $x \in E_{2n_0-1}$ (notation as before) we get

$$F'_p(x) = \sum_{j=1}^{m} \beta_j \mathbf{1}_{D_{t_j}}(n_0) a_{n_0} d'_{n_0}(x) = \beta_{j_0} a_{n_0} \tfrac{2^{2n_0}}{3} \neq 0,$$

and for λ-almost every $x \in E_{2n_0}$ we get

$$F'_p(x) = -\beta_{j_0} a_{n_0} \tfrac{2^{2n_0}}{3} \neq 0.$$

Considering that E_{2n_0} and E_{2n_0-1} are λ-dense it follows that the absolutely continuous function F_p is nowhere monotonic (and not identical to 0 on any non-degenerated interval).

(ii) Suppose that the assertion holds for all polynomials of degree $\leq r$ and suppose that p is of degree $r+1$. Then $F_p = p(f_{D_{t_1}}, \ldots, f_{D_{t_m}})$ is absolutely continuous and, letting ∂_j denote the partial derivative with respect to the j-th coordinate, F'_p can expressed as

$$F'_p(x) = \sum_{j=1}^{m} \partial_j p(f_{D_{t_1}}, \ldots, f_{D_{t_m}}) \cdot f'_{D_{t_j}}(x)$$

for λ-almost every $x \in [0,1]$. Denoting the smallest element integer j in $\{1,\ldots,m\}$ for which $\partial_j p$ is not identical to the zero-function by j_0 and choosing $n_0 \in D_{t_{j_0}} \cap \bigcap_{j\neq j_0}^{m} D^c_{t_j}$ then for λ-almost every $x \in E_{2n_0-1}$ we get

$$F'_p(x) = \partial_{j_0} p(f_{D_{t_1}}(x), \ldots, f_{D_{t_m}}(x)) a_{n_0} \tfrac{2^{2n_0}}{3},$$

and for λ-almost every $x \in E_{2n_0}$ we have

$$F'_p(x) = -\partial_{j_0} p(f_{D_{t_1}}(x), \ldots, f_{D_{t_m}}(x)) a_{n_0} \tfrac{2^{2n_0}}{3}.$$

According to the induction assumption, as a polynomial of degree $\leq r$ the function defined by $x \mapsto \partial_{j_0} p(f_{D_{t_1}}(x), \ldots, f_{D_{t_m}}(x))$ is not identically zero on any non-degenerated open interval, implying that F'_p is positive and negative on sets of positive measure in every non-degenerated open interval. The absolutely continuous function F_p therefore is nowhere monotonic and the proof is complete since $m \in \mathbb{N}$ was arbitrary. □

Remark 3.10. *A result that is related to Theorem 3.9 is the following one, that is due to Gámez et al.* [24]*: The set of differentiable functions on* $\mathbb{R}$ *that are nowhere monotone is* $\mathfrak{c}$*-lineable.*

Letting $h : [0, 1] \to [0, 1]$ denote an increasing singular homeomorphism, working with the singular probability measure μ_h induced by h, the sets $E'_n = h^{-1}(E_n), n \in \mathbb{N}$, and the functions $d'_n = d_n \circ h$ and $f'_D = f_D \circ h$ it is straightforward to translate the results derived before to the setting of singular functions and obtain the following theorem (a part of which has already been established in [9] using different techniques).

Theorem 3.11. *The family of all singular, nowhere monotonic functions on* $[0, 1]$ *is spaceable in* $(BV, \|\cdot\|_{BV})$ *and strongly* $\mathfrak{c}$*-algebrable.*

Remark 3.12. *Concerning Theorem 3.11, it is worth mentioning that it is not possible to replace the supporting space* $(BV, \|\cdot\|_{BV})$ *with* $(\mathcal{C}[0, 1], \|\cdot\|_\infty)$ *because of the celebrated Levine–Milman's theorem* [32] *asserting the non-existence of infinite dimensional closed vector subspaces in the latter space formed entirely by bounded variation functions.*

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