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On bivariate Archimedean copulas with fractal support

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Abstract: Due to their simple analytic form (bivariate) Archimedean copulas are usually viewed as very smooth and handy objects, which should distribute mass in a fairly regular and certainly not in a pathological way. Building upon recently established results on the Archimedean family and working with Iterated Function Systems with Probabilities (IFSP) we falsify this natural conjecture and derive the surprising result that for every $s \in [1, 2]$ there exists some bivariate Archimedean copula A_s fulfilling that the Hausdorff dimension of the support of A_s is exactly s.

Keywords: Copula, Doubly stochastic measure, Fractal, Singular function, Markov kernel

MSC: 62H20, 60E05, 28A80, 26A30

1 Introduction

Considering Lipschitz continuity and the fact that bivariate copulas are distributions functions (restricted to $[0, 1]^2$) of random vectors (X, Y) with X, Y being uniformly distributed on $[0, 1]$, it seems somehow natural to conjecture that copulas distribute their mass in a fairly regular way. Using Iterated Function Systems (IFSs) in 2005 Fredricks et al. (see [14]) falsified this very conjecture by constructing bivariate copulas with fractal support. In fact, the authors showed the existence of families $(A_r)_{r\in(0,1/2)}$ of bivariate copulas fulfilling the following property: for every $s \in (1, 2)$ there exists some $r_s \in (0, 1/2)$ such that the Hausdorff dimension of the support Z_{r_s} of A_{r_s} is exactly s; in other words: the smallest closed subset of $[0, 1]^2$ with full mass 1 has Hausdorff dimension s.

Since then various papers on copulas with fractal support have appeared in the literature, each of them underlining the fact that analytically nice objects (Lipschitz continuous, common marginals, etc.) like copulas may exhibit surprisingly irregular/pathological analytic behavior: In [27] we showed that the result by Fredricks et al. also holds for the subclass of so-called idempotent copulas (idempotent with respect to the star-product going back to Darsow et al. in [6] and corresponding to the standard composition of transition probabilities well known from the Markov chain setting) and generalized the IFS construction to arbitrary dimensions $d \geq 3$. Families $(A_r)_{r\in(0,1/2)}$ of copulas with fractal support were also studied by de Amo et al. in [1] and in [2], moments of these copulas were calculated in [3]. Some exotic properties of homeomorphisms between fractal supports of copulas were studied in [4], an alternative proof for the result by Fredricks et al. via so-called spatially homogeneous copulas was provided in [9], and Kendall's τ of mutually completely dependent copulas with self-similar support was determined in [13].

While each of the afore-mentioned contributions illustrates that the family $\mathcal{C}$ of all bivariate copulas contains analytically highly irregular elements, one might still conjecture that standard, commonly used subclasses like the bivariate Extreme-Value and the bivariate Archimedean family do not allow for such pathological behavior. The results given in [22, 28] verify this conjecture in the Extreme-Value setting - in this case the support of the copula has integer Hausdorff dimension 1 or 2 and coincides with the area between the graphs of two non-

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decreasing functions. As we will show in this note, however, in the Archimedean setting it is indeed possible to establish a result analogous to the one by Fredrick et al. In fact, for every fixed $s \in [1, 2]$ we will construct an Archimedean copula A_s whose support has Hausdorff dimension s, i.e., $\dim_H(\mathrm{supp}(A_s) = s$ holds. Contrary to the afore-mentioned papers we do not work with the IFS approach directly in the class $\mathcal{C}$ of bivariate copulas but use it to construct Archimedean generators φ of sufficient irregularity, which is then shown to propagate to the corresponding Archimedean copula A_φ. As a nice by-product of the studied construction we derive an analogous result for the (measure corresponding to the) Kendall distribution function $F_{A_\varphi}^{Kendall}$, i.e., we show that for every $r \in [0, 1]$ there exists some copula A_r whose Kendall distribution function has a support with Hausdorff dimension r.

The remainder of this note is organized as follows: Section 2 gathers notation and preliminaries, Section 3 some auxiliary results on Cantor functions needed in the sequel. All main results are presented in Section 4. Several graphics and an example corresponding to the classical middle third Cantor set illustrate the chosen procedures and underlying ideas.

2 Notation and preliminaries

For every metric space (Ω, ρ) the Borel σ-field on Ω will be denoted by $\mathcal{B}(\Omega)$, the family of all probability measures on $\mathcal{B}(\Omega)$ by $\mathcal{P}(\Omega)$. The support of a measure $\vartheta \in \mathcal{P}(\Omega)$, defined as the set of all points $x \in \Omega$, fulfilling that every open ball $B(x, r)$ of radius $r > 0$ around x fulfills $\vartheta(B(x, r)) > 0$, will be denoted by $\mathrm{supp}(\vartheta)$. It is well-known (see [25]) that $\mathrm{supp}(\vartheta)$ is closed.
As already mentioned before, $\mathcal{C}$ will denote the family of all two-dimensional *copulas*, i.e., the family of distribution functions (restricted to $[0, 1]^2$) of random vectors (X, Y) on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, fulfilling that the marginal distributions $\mathbb{P}^X, \mathbb{P}^Y$ coincide with the Lebesgue measure λ on $[0, 1]$. Letting d_∞ denote the uniform metric on $\mathcal{C}$ it is well known that $(\mathcal{C}, d_\infty)$ is a compact metric space. M will denote the minimum copula, Π the product copula and W the lower Fréchet-Hoeffding bound. For every $C \in \mathcal{C}$ the corresponding doubly stochastic measure will be denoted by μ_C and $\mathcal{P}_\mathcal{C} \subset \mathcal{P}([0, 1]^2)$ will denote the family of all *doubly stochastic measures*. By definition, the support of a copula C is the support of its corresponding doubly stochastic measure μ_C. Considering compactness of $[0, 1]^2$ the support of every copula is (as closed subset of a compact set) compact. For general background on copulas and doubly stochastic measure we refer to the textbooks [8, 23].
A *Markov kernel* from $\mathbb{R}$ to $\mathcal{B}(\mathbb{R})$ is a mapping $K : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ such that $x \mapsto K(x, B)$ is measurable for every fixed $B \in \mathcal{B}(\mathbb{R})$ and $B \mapsto K(x, B)$ is a probability measure for every fixed $x \in \mathbb{R}$. Given real-valued random variables X, Y on a joint probability space $(\Omega, \mathcal{A}, \mathbb{P})$, then a Markov kernel $K : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ is called a *regular conditional distribution of* Y *given* X if for every $B \in \mathcal{B}(\mathbb{R})$

$$K(X(\omega), B) = \mathbb{E}(\mathbf{1}_B \circ Y | X)(\omega) \tag{1}$$

holds for $\mathbb{P}$-almost every $\omega \in \Omega$. It is well known that for each pair (X, Y) of real-valued random variables a regular conditional distribution $K(\cdot, \cdot)$ of Y given X exists, that $K(x, \cdot)$ is unique $\mathbb{P}^X$-almost everywhere (i.e., unique for $\mathbb{P}^X$-almost all $x \in \mathbb{R}$) and that $K(\cdot, \cdot)$ only depends on the joint distribution $\mathbb{P}^{(X,Y)}$ of (X, Y). Hence, given $C \in \mathcal{C}$ we will denote (a version of) the regular conditional distribution of Y given X by $K_C(\cdot, \cdot)$, view it directly as a function mapping $[0, 1] \times \mathcal{B}([0, 1]) \to [0, 1]$ and refer to $K_C(\cdot, \cdot)$ simply as *regular conditional distribution of* C or as *Markov kernel of* C. Note that for every $C \in \mathcal{C}$, its Markov kernel $K_C(\cdot, \cdot)$, and every Borel set $G \in \mathcal{B}([0, 1]^2)$ we have ($G_x := \{y \in [0, 1] : (x, y) \in G\}$ denoting the x-section of G for every $x \in [0, 1]$)

$$\int_{[0,1]} K_C(x, G_x)\, d\lambda(x) = \mu_C(G). \tag{2}$$

As a special case the latter yields that

$$\int_{[0,1]} K_C(x, F)\, d\lambda(x) = \lambda(F) \tag{3}$$

holds for every $F \in \mathcal{B}([0,1])$. On the other hand, every Markov kernel $K : [0,1] \times \mathcal{B}([0,1]) \to [0,1]$ fulfilling equation (3) induces a unique element $\mu \in \mathcal{P}_C$ via equation (2). For every $C \in \mathcal{C}$ and $x \in [0,1]$ the function $y \mapsto G_x^C(y) := K_C(x, [0, y])$ will be called *conditional distribution function of C at x*. For more details and properties of conditional expectation, regular conditional distributions, and disintegration we refer to the excellent textbooks [16, 19].

Archimedean copulas can be expressed via generators $\varphi : [0,1] \to [0,\infty]$ (see, e.g., [12, 23]) or, equivalently, via generators $\psi : [0,1] \to [0,\infty)$ (see, e.g., [18, 21]). Here we use the former approach since it facilitates our construction. Following [23], a function $\varphi : [0,1] \to [0,\infty]$ is called *generator* if φ is convex on $(0,1]$, continuous and strictly decreasing on $[0,1]$ and fulfills $\varphi(1) = 0$. A generator φ is called *strict* if $\varphi(0) = \infty$ holds; in case of $\varphi(0) < \infty$ we will refer to φ as *non-strict*. For every generator φ we will let $\psi : [0,\infty) \to [0,1]$ denote its pseudo-inverse, defined by

$$\psi(t) = \begin{cases} \varphi^{-1}(t) & \text{if } t \in [0, \varphi(0)) \\ 0 & \text{if } t \geq \varphi(0). \end{cases}$$

To simplify notation in what follows we will work with the convention $\psi(\infty) := 0$. If φ is strict then obviously ψ coincides with the standard inverse. Every (strict or non-strict) generator φ induces a symmetric copula A_φ via

$$A_\varphi(x, y) = \psi\big(\varphi(x) + \varphi(y)\big), \qquad x, y \in [0,1],$$

to which we will refer as the (strict or non-strict) *Archimedean copula* induced by φ. The family of all bivariate Archimedean copulas will be denoted by $\mathcal{C}_{ar}$. In what follows we will let $[A_\varphi]^t$ denote the lower t-cut of A_φ, i.e.,

$$[A_\varphi]^t = \Big\{(x, y) \in [0,1]^2 : A_\varphi(x, y) \leq t\Big\}.$$

According to [23] for every Archimedean copula A_φ the level set $L_t := \{(x, y) \in [0,1]^2 : A_\varphi(x, y) = t\}$ is a convex curve for every $t \in (0,1]$. For $t = 0$ we obviously have $L_0 = (\{0\} \times [0,1]) \cup ([0,1] \times \{0\})$ if φ is strict whereas L_0 has positive area if φ is non-strict. Defining the function $f^t : [t,1] \to [0,1]$ for $t \in (0,1]$ by

$$f^t(x) := \psi(\varphi(t) - \varphi(x)) \tag{4}$$

we obviously have

$$\Gamma(f^t) := \big\{(x, f^t(x)) : x \in [t,1]\big\} = L_t \tag{5}$$

for every $t \in (0,1]$, i.e., the graph $\Gamma(f^t)$ of f^t coincides with the level curve L_t. For $t = 0$ we define f^0 analogous to equation (4) for $x > 0$ (with the convention $\infty - a = \infty$ for every finite a) and set $f^0(0) = 1$. If φ is non-strict $L_0 = \{(x, y) \in [0,1]^2 : y \leq f^0(x)\}$ holds, i.e., the graph of f^0 is the upper bound of L_0. It is straightforward to verify that for every $A_\varphi \in \mathcal{C}_{ar}$ and every $x \in (0,1]$ the function $y \mapsto A_\varphi(x, y)$ is strictly increasing on $[f^0(x), 1]$.

Before providing an explicit expression of the Markov kernel of an Archimdean copula we first recall some analytic properties of generators that will be used in the sequel: For every generator $\varphi : [0,1] \to [0,\infty]$ we will let $D^+\varphi(x)$ ($D^-\varphi(x)$) denote the right-hand (left-hand) derivative of φ at $x \in (0,1)$. Convexity of φ implies that $D^+\varphi(x) = D^-\varphi(x)$ holds for all but at most countably many $x \in (0,1)$, i.e., φ is differentiable outside a countable subset of $(0,1)$, and that $D^+\varphi$ is non decreasing and right-continuous while $D^-\varphi$ is non decreasing and left continuous (see [17, 24]). Setting $D^+\varphi(0) = -\infty$ in case of strict φ as well as $D^+\varphi(1) = 0$ (for strict and non-strict ones) allows to view $D^+\varphi$ as non-decreasing and right-continuous function on the full unit interval $[0,1]$. Additionally (again see [17, 24]), we have $D^-\varphi(x) = D^+\varphi(x-)$ for every $x \in (0,1)$.

To simplify notation, for every $a \in \mathbb{R}$ expressions of the from $\frac{a}{-\infty}$ will be interpreted as zero in what follows. If φ is non-strict, then according to [12] $K_{A_\varphi}(\cdot,\cdot)$, defined by

$$K_{A_\varphi}(x,[0,y]) := \begin{cases} 1 & \text{if } x \in \{0,1\} \\ \frac{D^+\varphi(x)}{(D^+\varphi)(A_\varphi(x,y))} & \text{if } x \in (0,1) \text{ and } y \geq f^0(x) \\ 0 & \text{if } x \in (0,1) \text{ and } y < f^0(x) \end{cases} \tag{6}$$

is a Markov kernel of A_φ. In the strict case, equation (6) remains valid but the last line becomes obsolete since in this case $f^0(x) = 0$ holds for every $x \in (0,1]$. For every generator φ (again see [12, 23]) the following identity holds for the mass of the level-sets $L_t = \Gamma(f^t)$:

$$\mu_{A_\varphi}(L_t) = -\frac{\varphi(t)}{D^+\varphi(t)} + \frac{\varphi(t)}{D^+\varphi(t-)} = -\frac{\varphi(t)}{D^+\varphi(t)} + \frac{\varphi(t)}{D^-\varphi(t)}, \quad t \in (0,1). \tag{7}$$

Moreover, in the strict case $\mu_{A_\varphi}(L_0) = 0$ holds and for non-strict φ we have $\mu_{A_\varphi}(L_0) = -\frac{\varphi(0)}{D^+\varphi(0)}$, so in the latter case L_0 may or may not have mass depending on whether $D^+\varphi(0)$ is unbounded or not. Notice that formula (7) offers a nice geometric interpretation (see [12, Figure 1]): $\mu_{A_\varphi}(L_t)$ coincides with the length of the line segment on the x-axis generated by left- and right-hand tangents of φ at t.

As shown in [12, 23], the Kendall distribution function $F^{Kendall}_{A_\varphi}$ of a non-strict Archimedean copula A_φ is given by

$$F^{Kendall}_{A_\varphi}(t) = \mu_{A_\varphi}\left([A_\varphi]^t\right) = \begin{cases} -\frac{\varphi(0)}{D^+\varphi(0)} & \text{if } t = 0 \\ t - \frac{\varphi(t)}{D^+\varphi(t)} & \text{if } t \in (0,1], \end{cases} \tag{8}$$

and in the strict case we have

$$F^{Kendall}_{A_\varphi}(t) = \mu_{A_\varphi}\left([A_\varphi]^t\right) = \begin{cases} 0 & \text{if } t = 0 \\ t - \frac{\varphi(t)}{D^+\varphi(t)} & \text{if } t \in (0,1]. \end{cases}$$

We will directly use these expressions in the next section in order to show that the probability measure κ_{A_φ} corresponding to the Kendall distribution function has fractal support.

As last key component we recall the definition of an Iterated Function System (IFS for short) and some main results about IFSs (for more details see [5, 10, 11, 20]). Suppose for the following that (Ω, ρ) is a compact metric space and let δ_H denote the Hausdorff metric on the family $\mathcal{K}(\Omega)$ of all non-empty compact subsets of Ω. A mapping $w : \Omega \to \Omega$ is called *contraction* if there exists a constant $L < 1$ such that $\rho(w(x), w(y)) \leq L\rho(x,y)$ holds for all $x, y \in \Omega$. A family $(w_l)_{l=1}^N$ of $N \geq 2$ contractions on Ω is called *Iterated Function System* and will be denoted by $\{\Omega, (w_l)_{l=1}^N\}$. An IFS together with a vector $(p_l)_{l=1}^N \in (0,1]^N$ fulfilling $\sum_{l=1}^N p_l = 1$ is called *Iterated Function System with Probabilities* (IFSP for short) and will be denoted by $\{\Omega, (w_l)_{l=1}^N, (p_l)_{l=1}^N\}$. Every IFSP induces the so-called *Hutchinson operator* $\mathcal{H} : \mathcal{K}(\Omega) \to \mathcal{K}(\Omega)$, defined by

$$\mathcal{H}(Z) := \bigcup_{l=1}^N w_l(Z). \tag{9}$$

It can be shown (see [5, 11, 20]) that $\mathcal{H}$ is a contraction on the compact metric space $(\mathcal{K}(\Omega), \delta_H)$, so Banach's Fixed Point theorem implies the existence of a unique, globally attractive fixed point $Z^\star \in \mathcal{K}(\Omega)$ of $\mathcal{H}$. Hence, for every $R \in \mathcal{K}(\Omega)$, we have

$$\lim_{n\to\infty} \delta_H\left(\mathcal{H}^n(R), Z^\star\right) = 0.$$

The *attractor* $Z^\star$ will be called *self-similar* if all contractions in the IFS are similarities. An IFS $\{\Omega, (w_l)_{l=1}^N\}$ is called *totally disconnected* (or disjoint) if the sets $w_1(Z^\star), w_2(Z^\star), \ldots, w_N(Z^\star)$ are pairwise disjoint.

Additionally to the operator $\mathcal{H}$ every IFSP also induces a (Markov) operator $\mathcal{V} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$, defined by ($\vartheta^{w_i}$ denoting the push-forward of ϑ via w_i)

$$\mathcal{V}(\mu) := \sum_{i=1}^N p_i\, \vartheta^{w_i}. \tag{10}$$

The so-called *Hutchison metric* h (sometimes also called Kantorovich or Wasserstein metric) on $\mathcal{P}(\Omega)$ is defined by

$$h(\vartheta, \nu) := \sup \Big\{ \int\limits_\Omega f \, d\vartheta - \int\limits_\Omega f \, d\nu : \, f \in Lip_1(\Omega, \mathbb{R}) \Big\}, \tag{11}$$

where $Lip_1(\Omega, \mathbb{R})$ is the class of all non-expanding functions $f : \Omega \to \mathbb{R}$, i.e., functions fulfilling $|f(x) - f(y)| \leq \rho(x, y)$ for all $x, y \in \Omega$. It is not difficult to show that $\mathcal{V}$ is a contraction on $(\mathcal{P}(\Omega), h)$, that h is a metrization of the topology of weak convergence on $\mathcal{P}(\Omega)$ and that $(\mathcal{P}(\Omega), h)$ is a compact metric space (see [5, 7]). Consequently, again by Banach's Fixed Point theorem, it follows that there is a unique, globally attractive fixed point $\vartheta^\star \in \mathcal{P}(\Omega)$ of $\mathcal{V}$, i.e., for every $\nu \in \mathcal{P}(\Omega)$ we have

$$\lim_{n \to \infty} h\big(\mathcal{V}^n(\nu), \vartheta^\star\big) = 0.$$

The fixed point $\vartheta^\star$ will be called *invariant measure*. It is well known that the support $\mathrm{supp}(\vartheta^\star)$ of $\vartheta^\star$ is exactly the attractor $Z^\star$ (again see [5, 10, 20, 11]). The measure $\vartheta^\star$ will be called *self-similar* if $Z^\star$ is self-similar, i.e., if all contractions in the IFSP are similarities.

Attractors of IFSs are strongly interrelated with symbolical dynamical systems via the so-called *address map* (see [5, 20]): For every $N \in \mathbb{N}$ the *code space of N symbols* will be denoted by Σ_N, i.e.

$$\Sigma_N := \{1, 2, \ldots, N\}^{\mathbb{N}} = \big\{(k_i)_{i \in \mathbb{N}} : 1 \leq k_i \leq N \; \forall i \in \mathbb{N}\big\}.$$

To simplify notation in what follows we will write $\mathbf{k} = (k_1, k_2, \ldots)$ for element of Σ_N. Moreover, the (left-) shift operator on Σ_N will be denoted by σ, i.e., $\sigma((k_1, k_2, \ldots)) = (k_2, k_3, \ldots)$. Defining a metric m on Σ_N by setting

$$m(\mathbf{k}, \mathbf{l}) := \begin{cases} 0 & \text{if } \mathbf{k} = \mathbf{l} \\ 2^{1 - \min\{i : k_i \neq l_i\}} & \text{if } \mathbf{k} \neq \mathbf{l}, \end{cases}$$

it is straightforward to verify that (Σ_N, m) is a compact ultrametric space and that m is a metrization of the product topology.

Suppose now that $\{\Omega, (w_l)_{l=1}^N\}$ is an IFS with attractor $Z^\star$, fix an arbitrary $x \in \Omega$ and define the *address map* $G : \Sigma_N \to \Omega$ by

$$G(\mathbf{k}) := \lim_{n \to \infty} w_{k_1} \circ w_{k_2} \circ \cdots \circ w_{k_n}(x), \tag{12}$$

then (see [20]) $G(\mathbf{k})$ is independent of x, $G : \Sigma_N \to \Omega$ is Lipschitz continuous and $G(\Sigma_N) = Z^\star$. Furthermore G is injective (and hence a homeomorphism) if, and only if the IFS is totally disconnected. Given $z \in Z^\star$ every element of the preimage $G^{-1}(\{z\})$ will be called *address* of z. Considering a IFSP $\{\Omega, (w_l)_{l=1}^N, (p_l)_{l=1}^N\}$ with attractor $Z^\star$ and invariant measure $\mu^\star$ we can further define a probability measure P on $\mathcal{B}(\Sigma_N)$ by setting

$$P\Big(\big\{\mathbf{k} \in \Sigma_N : k_1 = i_1, k_2 = i_2, \ldots, k_m = i_m\big\}\Big) = \prod_{j=1}^m p_{i_j} \tag{13}$$

and extending in the standard way to full $\mathcal{B}(\Sigma_N)$. According to [20] $\mu^\star$ is the push-forward of P via the address map, i.e., $P^G(B) = P(G^{-1}(B)) = \mu^\star(B)$ holds for each $B \in \mathcal{B}(Z^\star)$.

3 Auxiliary results on Cantor functions

Since for the construction of Archimedean copulas with fractal support we will work with Cantor functions, we recall their construction via IFSs (only consisting of two functions) and then derive some properties needed in the sequel. For every $r \in (0, \frac{1}{2})$ let the similarities $w_1^r, w_2^r : [0,1] \to [0,1]$ be defined by

$$w_1^r(x) = rx, \quad w_2^r(x) = 1 - rx, \tag{14}$$

set $p_1 = p_2 = \frac{1}{2}$, consider the totally disconnected IFSP $\{[0,1], (w_l^r)_{l=1}^2, (p_l)_{l=1}^2\}$ and denote the corresponding Hutchinson and Markov operator by $\mathcal{H}_r$ and $\mathcal{V}_r$, respectively. As mentioned before, both $\mathcal{H}_r$ and $\mathcal{V}_r$ have unique fixed points $Z_r^\star \in \mathcal{K}([0,1])$ and $\vartheta_r^\star \in \mathcal{P}([0,1])$, respectively, which are linked via

$$\text{supp}(\vartheta_r^\star) = Z_r^\star.$$

Obviously $Z_r^\star$ and $\vartheta_r^\star$ are self-similar, so (see [5, 11]) the Hausdorff dimension $\dim_H(Z_r^\star)$ of $Z_r^\star$ is the unique solution s of the equation $2r^s = 1$, i.e.,

$$\dim_H(Z_r^\star) = -\frac{\log(2)}{\log(r)} \in (0,1) \tag{15}$$

holds. Defining $\iota : (0, \frac{1}{2}) \to (0,1)$ by $\iota(r) = -\frac{\log(2)}{\log(r)}$, obviously ι is a bijection. Considering that the IFS $\{[0,1], (w_l^r)_{l=1}^2\}$ is totally disconnected, the address maps G, defined according to equation (12), is a homeomorphism. Hence, using the fact that the product measure P on Σ_2 has no atoms, it follows that the invariant measures $\vartheta_r^\star$ does not have any point masses. Letting $G_r^\star$ denote the distribution function (restricted to $[0,1]$) corresponding to $\vartheta_r^\star$ this shows that $G_r^\star : [0,1] \to [0,1]$ is continuous. The construction of $Z_r^\star$ implies that $[0,1] \setminus Z_r^\star$ can be expressed as

$$[0,1] \setminus Z_r^\star = \bigcup_{i=1}^\infty J_i,$$

with $J_1, J_2, \ldots$ denoting pairwise disjoint, non-degenerated open intervals, given by

$$\begin{aligned} J_1 &= (r, 1-r), \\ J_2 &= w_1^r(J_1) = (r^2, r(1-r)), \\ J_3 &= w_2^r(J_1) = (1 - r(1-r),\ 1 - r^2), \\ J_4 &= w_1^r \circ w_1^r(J_1) = (r^3, r^2(1-r)), \\ &\ldots \end{aligned} \tag{16}$$

Notice that using the Hutchinson operator we obviously have

$$\bigcup_{i=1}^\infty J_i = \bigcup_{i=1}^\infty \mathcal{H}_r^i\left((r, 1-r)\right).$$

Considering that the IFSP construction of $\vartheta_r^\star$ implies that $G_r^\star$ is constant on each J_i, it follows immediately that $G_r^\star$ is a singular distribution function, i.e., $G_r^\star$ is a continuous distribution function fulfilling that the derivative $(G_r^\star)'$ is identical to zero λ-almost everywhere in $[0,1]$. Moreover, the property $\text{supp}(\vartheta^\star) = Z_r^\star$ implies that for every $x \in [0,1]$ the following equivalence holds (extend $G_r^\star$ to $\mathbb{R}$ by setting $G(x) = 0$ for $x < 0$ and $G(x) = 1$ for $x > 0$ to assure that all expressions are well-defined):

$$x \in Z_r^\star \quad \Longleftrightarrow \quad G_r^\star(x + \Delta) - G_r^\star(x - \Delta) > 0 \text{ for every } \Delta > 0. \tag{17}$$

For confirming that the function φ_r considered in the next section is indeed an Archimedean generator we need the property

$$\int_{[0,1]} G_r^\star(x) d\lambda(x) = \frac{1}{2}, \tag{18}$$

which either follows geometrically by using symmetry (the endo- and the hypergraph of $G_r^\star$ have the same area) or, alternatively, as follows: the measure $\vartheta_r^\star$ obviously is symmetric in the sense that $(\vartheta_r^\star)^{1-id} = \vartheta_r^\star$ holds - in fact for every measure $\vartheta \in \mathcal{P}([0,1])$ the measure $\mathcal{V}_r(\vartheta)$ has this property. Using change of coordinates this shows

$$\begin{aligned}\int_{[0,1]} t\, d\vartheta_r^\star(t) &= \int_{[0,1]} t\, d(\vartheta_r^\star)^{1-id}(t) = \int_{[0,1]} (1-t)\, d\vartheta_r^\star(t) \\ &= 1 - \int_{[0,1]} t\, d\vartheta_r^\star(t),\end{aligned}$$

implying $\int_{[0,1]} t\, d\vartheta_r^\star(t) = \frac{1}{2}$. Finally, using the fact that for non-negative, integrable random variables X with distribution function F we have $\mathbb{E}(X) = \int_{[0,\infty)} (1-F(t))d\lambda(t)$ yields equation (18).

Example 3.1. For the case $r_0 = \frac{1}{3}$ the fixed point $Z_{r_0}^\star$ is the classical (middle third) Cantor set (see [11]) with Hausdorff dimension $\dim_H(Z_{r_0}^\star) = \frac{\log(2)}{\log(3)}$. The distribution function $G_{r_0}^\star$ is the classical Cantor function (a.k.a. devil's staircase). Figure 1 depicts an approximation of $G_{r_0}^\star$ - in fact, the gray line is the graph of the distribution function corresponding to the probability measure $\mathcal{V}_{r_0}^n(\lambda)$ for $n = 8$.

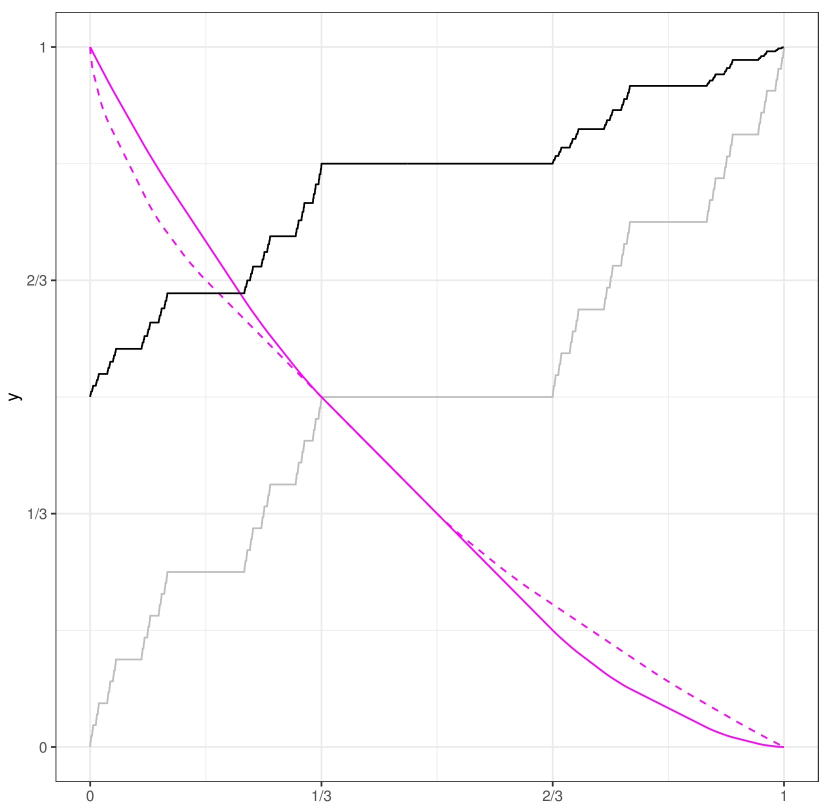

Fig. 1: Approximation of the classical (middle third) Cantor function $G_{r_0}^\star$ as considered in Example 3.1 and Example 4.3 (gray line). The black line depicts (an approximation of) the Kendall distribution function of the copula A_{r_0}, the solid magenta line is (an approximation of) the generator φ_{r_0}, the dashed line the corresponding pseudo-inverse ψ_{r_0}.

4 Constructing bivariate Archimedean copulas with fractal support

Let $r \in (0, \frac{1}{2})$ be arbitrary but fixed and define the function $\varphi_r : [0,1] \to \mathbb{R}$ by

$$\varphi_r(x) := \int_{[0,x]} (-2 + 2\,G_r^\star(t))\, d\lambda(t) + 1. \tag{19}$$

Then obviously we have $\varphi_r(0) = 1$ and, using equation (18), $\varphi_r(1) = 0$ follows. Moreover, considering that the integrand in equation (19) is negative on $[0,1)$ and non-decreasing on $[0,1]$, it follows that φ_r is convex (see [17, 24]) and strictly decreasing on $[0,1]$. Altogether, φ_r is a non-strict generator with right-hand derivative given by $D^+\varphi_r(t) = -2 + 2\,G_r^\star(t)$. The magenta line in Figure 1 depicts the generator φ_r for the case $r = \frac{1}{3}$, the dashed magenta line is the corresponding pseudo-inverse ψ_r.

Letting $A_r := A_{\varphi_r} \in \mathcal{C}_{ar}$ denote the induced Archimedean copula (see Figure 2 for the case $r = \frac{1}{3}$), using the results from Section 2 it follows that

$$\mu_{A_r}(L_0) = \mu_{A_r}(\Gamma(f^0)) = -\frac{\varphi_r(0)}{D^+\varphi_r(0)} = \frac{1}{2},$$

i.e., the copula A_r assigns half of its mass to the graph of f^0. Using continuity of $D^+\varphi_r$ and equation (7) all other level sets L_t carry no mass, i.e., $\mu_{A_r}(L_t) = 0$ holds for every $t \in (0,1]$. Moreover, the Kendall distribution function $F_{A_r}^{Kendall}$ fulfills $F_{A_r}^{Kendall}(0) = \frac{1}{2}$ as well as

$$F_{A_r}^{Kendall}(t) = t - \frac{\varphi_r(t)}{-2 + 2\,G_r^\star(t)}$$

for every $t \in (0,1]$.

We are now going to show that the Hausdorff dimension of the support of μ_{A_r} is given by $\mathrm{supp}(\mu_{A_r}) = 1 - \frac{\log(2)}{\log(r)}$ and the one of the support of the measure κ_{A_r} by $-\frac{\log(2)}{\log(r)}$. Doing so, we proceed in several steps formalized as lemmas and work with the sets $L_J \subseteq [0,1]^2$, defined by

$$L_J := \Big\{(x,y) \in [0,1]^2 : \; A_r(x,y) \in J\Big\}$$

for every interval $J \subseteq [0,1]$.

Lemma 4.1. *Letting $J_1, J_2, \dots$ denote the open intervals defined according to equation (16), the following identity holds:*

$$\mu_{A_r}\left(\bigcup_{i=1}^{\infty} L_{J_i}\right) = 0 = \kappa_{A_r}\left(\bigcup_{i=1}^{\infty} J_i\right).$$

Proof. Notice that each of the intervals J_i has the property that the function $G_r^\star$, hence the function $D^+\varphi_r$, is constant on J_i. As a direct consequence, writing $J_i = (a,b)$ it follows that $\varphi_r(b) = \varphi_r(a) + (b-a)D^+\varphi_r(a)$, from which we directly get

$$\begin{aligned} \mu_{A_r}(L_{(a,b]}) &= \kappa_{A_r}((a,b]) = F_{A_r}^{Kendall}(b) - F_{A_r}^{Kendall}(a) \\ &= b - \frac{\varphi_r(b)}{D^+\varphi_r(b)} - \left(a - \frac{\varphi_r(a)}{D^+\varphi_r(a)}\right) \\ &= b - a - \frac{\varphi_r(a) + (b-a)\varphi_r(a) - \varphi_r(a)}{\varphi_r(a)} \\ &= 0. \end{aligned}$$

Having this, using $L_{(a,b)} \subseteq L_{(a,b]}$ as well as σ-additivity of μ_{A_r} and κ_{A_r} yields the desired identity. □

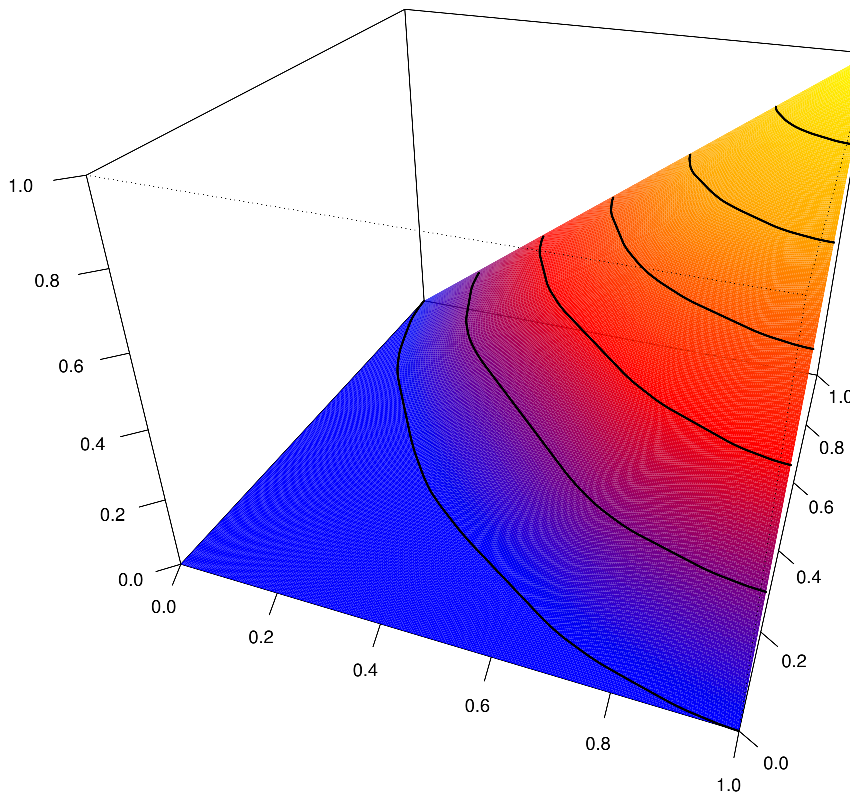

Fig. 2: 3d-plot of (an approximation of) the copula A_{r_0} considered in Example 3.1 and Example 4.3; the lines depict the graphs of the function f^t with $t \in \{0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}\}$.

Lemma 4.2. *The support supp*(μ_{A_r}) *of* μ_{A_r} *is given by*

$$supp(\mu_{A_r}) = \bigcup_{t \in Z_r^\star} L_t = \bigcup_{t \in Z_r^\star} \Gamma(f^t). \tag{20}$$

Moreover, the support of κ_{A_r} *coincides with* $Z_r^\star$.

Proof. We show that for every $t \in Z_r^\star$ and $x > t$ the point $(x, f^t(x)) \in (0,1)^2$ is an element of $\text{supp}(\mu_{A_r})$ and proceed as follows: Setting $I := (f^t(x) - \delta, f^t(x) + \delta] \subseteq [0,1]$ for sufficiently small $\delta > 0$, the Markov kernel $K_{A_r}(\cdot,\cdot)$, defined according to equation (6), fulfills

$$K_{A_r}(x, I) \quad = \quad \frac{D^+\varphi_r(x)}{D^+\varphi_r\left(A_r(x, f^t(x) + \delta)\right)} - \frac{D^+\varphi_r(x)}{D^+\varphi_r\left(A_r(x, f^t(x) - \delta)\right)}.$$

Considering $A_r(x, f^t(x) + \delta) > t$, $A_r(x, f^t(x) - \delta) < t$ and using equivalence (17)

$$D^+\varphi_r\Big(A_r(x, f^t(x) - \delta)\Big) < D^+\varphi_r\Big(A_r(x, f^t(x) + \delta)\Big)$$

follows, which directly yields $K_{A_r}(x, I) > 0$. Since $\delta > 0$ can be chosen arbitrarily small it follows that $f^t(x) \in \text{supp}(K_{A_r}(x,\cdot))$. Convexity of the function $f^t : [t,1] \to [t,1]$ implies that f^t is continuous, so using disintegration it follows immediately that every open square

$$S := (x - \Delta, x + \Delta) \times (f^t(x) - \Delta, f^t(x) + \Delta)$$

with sufficiently small $\Delta > 0$ fulfills $\mu_{A_r}(S) > 0$, implying $(x, f^t(x)) \in \text{supp}(A_r)$.
Proceeding in the same manner shows that for every $x > 0$ we have $(x, f^0(x)) \in \text{supp}(A_r)$. Therefore, using the fact that $\text{supp}(A_r)$ is compact (hence closed) it follows that

$$\text{supp}(\mu_{A_r}) \supseteq \bigcup_{t \in Z_r^\star} \Gamma(f^t).$$

Since the set $\bigcup_{i=1}^\infty L_{J_i}$ from Lemma 4.1 is as union of open sets open too, Lemma 4.1 implies

$$\text{supp}(\mu_{A_r}) \subseteq [0,1]^2 \setminus \bigcup_{i=1}^\infty L_{J_i} = \bigcup_{t \in Z_t^\star} \Gamma(f^t),$$

which completes the proof of the first assertion. The second assertion follows from equivalence (17). □

Example 4.3. For the case $r_0 = \frac{1}{3}$ considered in Example 3.1, the support $\text{supp}(\mu_{A_{r_0}})$ consists of uncountably many convex curves - the contour lines connecting the points $(t, 1)$ and $(1, t)$ with t being an element of the classical (middle third) Cantor set $G^\star_{r_0}$. In Lemma 4.4 we will show that the support of A_{r_0} has Hausdorff dimension $1 + \frac{\log(2)}{\log(3)}$. Figure 3 depicts an approximation of the support. Moreover, the black line in Figure 1 is an approximation of the corresponding Kendall distribution function $F^{Kendall}_{A_{r_0}}$ - notice that it obviously has the same plateaus but is not just a rescaled version of the Cantor function $G^\star_{r_0}$.

Lemma 4.4. *The support $\text{supp}(\mu_{A_r})$ of μ_{A_r} has Hausdorff dimension*

$$dim_H\,(supp(\mu_{A_r})) = 1 - \frac{\log(2)}{\log(r)}. \tag{21}$$

Proof. We will use the fact that bi-Lipschitz transformtions (see, e.g., [11]) preserve the Hausdorff dimension and proceed as follows: Define the sets Λ and T by

$$\Lambda := \bigcup_{t \in Z_r^\star} [t, 1] \times \{t\}, \quad T := \{(x, y) \in [0,1]^2 : y \leq x\}.$$

Then proceeding as with the support of μ_{A_r} before it is straightforward to show that Λ is compact, implying $\Lambda \in \mathcal{B}([0,1]^2)$. Letting the transformation $h : T \to [0,1]^2$ be defined by

$$h(x, t) := (x, \psi(\varphi(t) - \varphi(x))) = (x, f^t(x)),$$

obviously h maps Λ to $\text{supp}(\mu_{A_r})$, i.e., we have $h(\Lambda) = \text{supp}(\mu_{A_r})$. It is easy to verify that h is not bi-Lipschitz on the set T - in fact, the function f^t has unbounded derivative to the right of t and arbitrary small derivative close to 1. Considering, however, the triangle $T_n \subset T$, defined as the convex hull of the three points

$$E_n^1 = \left(\frac{1}{2 \cdot 3^n}, 0\right), \ E_n^2 := \left(1 - \frac{1}{2 \cdot 3^n}, 0\right), \ E_n^3 := \left(1 - \frac{1}{2 \cdot 3^n}, 1 - \frac{1}{3^n}\right)$$

for every $2 \leq n \in \mathbb{N}$, using the fact that both the derivative of φ_r and ψ_r is bounded from above and from below on every interval of the form $[a, b] \subseteq (0, 1)$, it is straightforward to show that h is indeed bi-Lipschitz on every T_n. As a direct consequence (see [11]) the Hausdorff dimension of the set $\Lambda \cap T_n$ and the Hausdorff dimension of $h(\Lambda \cap T_n)$ coincide, following [15] we have

$$\dim_H(\Lambda \cap T_n) = 1 - \frac{\log(2)}{\log(r)}.$$

Using countable stability of the Hausdorff dimension (again see [11]) therefore yields

$$\dim_H\left(\bigcup_{n=2}^\infty h(\Lambda \cap T_n)\right) = \sup_{n \geq 2} \dim_H(h(\Lambda \cap T_n)) = 1 - \frac{\log(2)}{\log(r)}.$$

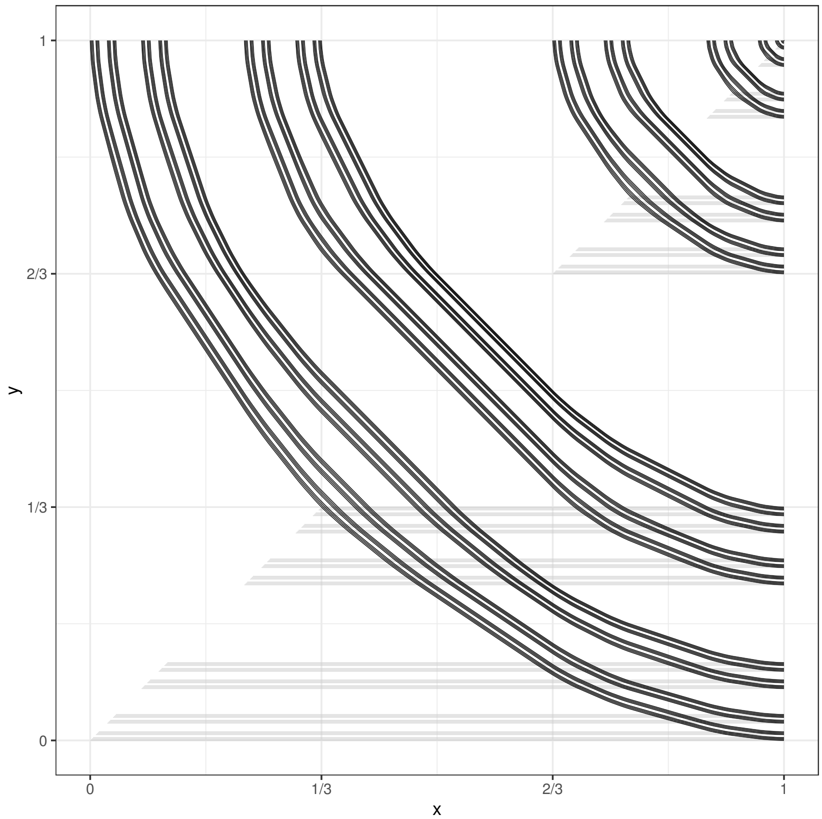

Fig. 3: Approximation of the support of the copula A_{r_0} as considered in Example 3.1 and Example 4.3 (black curves). The set Λ used in the proof of Lemma 4.4 (light gray).

Having that, considering that the sets $\{1\} \times Z_r^\star$ and $Z_r^\star \times \{1\}$ both have Hausdorff dimension $-\frac{\log(2)}{\log(r)}$ it altogether follows that

$$\begin{aligned} \dim_H(\mathrm{supp}(\mu_{A_r})) &= \max\left(\dim_H\left(\bigcup_{n=2}^{\infty} h(\Lambda \cap T_n)\right), -\frac{\log(2)}{\log(r)}\right) \\ &= 1 - \frac{\log(2)}{\log(r)}, \end{aligned}$$

which completes the proof. □

Summing up, we can finally formulate and prove our main result:

Theorem 4.5. *For every $s \in [1,2]$ there exists some bivariate Archimedean copula A with the following properties:*
1. $dim_H(supp(\mu_A)) = s$.
2. $dim_H(supp(\kappa_A)) = s - 1$.

Proof. Considering the fact that the afore-mentioned mapping $\iota : (0, \frac{1}{2}) \to (0,1)$, given by $\iota(r) = -\frac{\log(2)}{\log(r)}$ is surjective, using Lemma 4.2 and Lemma 4.4 immediately yields both assertions for the case $s \in (1,2)$. The remaining cases $s = 1$ and $s = 2$ are, however, trivial: for $s = 1$ consider $W \in \mathcal{C}_{ar}$, whose support is a straight line and for $s = 2$ use $\Pi \in \mathcal{C}_{ar}$, whose support is $[0,1]^2$. □

Remark 4.6. Recently established results focusing on the interplay of Archimedean copulas and so-called Williamson measures (see [18]) allow for alternative ways to construct Archimdean copulas with fractal support. One may, for instance, start with the classical Cantor measure $\vartheta^\star_{r_0}$ as Williamson measure and consider the (pseudo-inverse of the) generator $\psi = \mathcal{W}_2(\vartheta^\star_{r_0})$, where $\mathcal{W}_2$ denotes the Williamson transform in dimension $d = 2$. We opted for the approach via φ_r since in this case calculating the Hausdorff dimension of μ_{A_r} is much simpler.

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