On differentiability and mass distributions of multivariate Archimedean copulas

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Abstract

Copulas, in particular Archimedean copulas are commonly viewed as analytically nice and regular objects. Motivated by a recently established result stating that the first partial derivatives of bivariate copulas can exhibit surprisingly pathological behavior, we focus on the class of d-dimensional Archimedean copulas denoted by $\mathcal{C}_{ar}^d$ and show that partial derivatives of order $(d-1)$ can be surprisingly irregular as well. In fact, we prove the existence of Archimedean copulas $C \in \mathcal{C}_{ar}^d$ whose $(d-1)$-st order partial derivatives are pathological in the sense that for almost every $\mathbf{x} \in [0,1]^{d-1}$ the derivative $\partial_1 ... \partial_{d-1} C(\mathbf{x}, y)$ does not exist on a dense set of $y \in (0,1)$.

Since the existence of mixed partial derivatives of order $(d-1)$ of a copula C is closely related to the existence of a discrete component, we also study mass distributions of Archimedean copulas. Building upon the interplay between Archimedean copulas and so-called Williamson measures we show that absolute continuity, discreteness and singularity of the Williamson measure propagates to the associated Archimedean copula and vice versa. Moreover, we prove the fact that the sub-family of $\mathcal{C}_{ar}^d$ consisting of copulas whose absolutely continuous, discrete and singular component have full support is dense in $\mathcal{C}_{ar}^d$.

Finally, viewing $\mathcal{C}_{ar}^d$ in the light of Baire categories, we show that, in contrast to the space of bivariate copulas, a topologically typical d-dimensional Archimedean copula C is not absolutely continuous but has degenerate discrete component, implying that pathological elements are rare in $\mathcal{C}_{ar}^d$.

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1. Introduction

Due to their simple algebraic structure and their relevance in various applications, e.g., in finance and hydrology [7, 15, 22], Archimedean copulas gained increasing popularity over the last decades. Considering a so-called Archimedean generator $\psi\colon [0,\infty) \to [0,1] =: \mathbb{I}$ and its quasi-inverse function φ (and assuming that the generator ψ is sufficiently monotone/regular), setting

$$C_\psi(x_1, ..., x_d) := \psi(\varphi(x_1) + ... + \varphi(x_d)).$$

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defines a d-dimensional Archimedean copula C_ψ. Building upon the afore-mentioned regularity of the generator ψ it is known (see, e.g., [14, Theorem A1]) that for an arbitrary d-dimensional Archimedean copula C and every $y \in (0,1)$ the partial derivative $\partial_1 ... \partial_{d-1} C(\mathbf{x}, y)$ exists for λ_{d-1}-almost every $\mathbf{x} := (x_1, ..., x_{d-1}) \in \mathbb{I}^{d-1}$ (whereby λ_{d-1} denotes the $(d-1)$-dimensional Lebesgue measure and $\mathbb{I}$ the unit interval). However, as pointed out in the bivariate setting in [3], derivatives of copulas have to be handled with care - the main objective of this paper is to show that this statement remains true for the class $\mathcal{C}_{ar}^d$ of all d-dimensional Archimedean copulas.
Motivated by the results in [3], where it was shown that in the whole family $\mathcal{C}^2$ of bivariate copulas as well as in the family of bivariate extreme-value copulas there exist elements C with the property that the first partial derivative $\partial_1 C(\cdot, \cdot)$ does not exist on a dense set, we here establish a similar result for the family of d-dimensional Archimedean copulas and show the existence of an Archimedean copula C with the following property: there exists some Borel set $\Lambda \subseteq \mathbb{I}^{d-1}$ with $\mu_{C^{1:d-1}}(\Lambda) > 0$ (whereby $\mu_{C^{1:d-1}}$ is the $(d-1)$-stochastic measure corresponding to the marginal copula $C^{1:d-1}$ of the first $d-1$ coordinates) such that for λ_{d-1}-almost every $\mathbf{x} \in \mathbb{I}^{d-1}$ the mixed partial derivative $\partial_1 ... \partial_{d-1} C(\mathbf{x}, y)$ does not exist for a dense set of $y \in (0,1)$. More surprisingly, the family of all copulas exhibiting the afore-mentioned pathological behavior is still comparably large in the sense that it is dense in $\mathcal{C}_{ar}^d$ with respect to the standard uniform metric d_∞.
Building upon the fact that (for arbitrary dimension $d \geq 2$) multivariate Archimedean copulas C can be characterized in terms of so-called Williamson measures γ (probability measures γ on $(0, \infty)$, see [21]), we first derive various results on the interplay between the copula and its Williamson measure, which provide the basis for constructing copulas with afore-mentioned pathological differentiability properties. In particular, we show that the Williamson measure γ has a non-degenerate absolutely continuous/discrete/singular (singular in the sense that it has no point masses and its corresponding distribution function F_γ has derivative 0 λ-almost everywhere) component if, and only if the corresponding Archimedean copula C has non-degenerate absolutely continuous/discrete/singular component (in a sense specified in Section 2.2). Moreover we prove that $C \in \mathcal{C}_{ar}^d$ has full support if, and only if the corresponding Williamson measure γ has full support and then show that the same holds for the absolutely continuous, the discrete and the singular component. Building upon these results we are able to prove the surprising fact that the subfamily of $\mathcal{C}_{ar}^d$ consisting of copulas whose absolutely continuous, discrete and singular component, respectively, have full support is dense in $\mathcal{C}_{ar}^d$.
One question that arises naturally is, whether the afore-mentioned subfamilies are considered topologically 'small' or 'large', indicating whether their elements represent atypical or typical elements of $\mathcal{C}_{ar}^d$. Utilizing Baire categories (see, e.g., [23]), topology provides a natural framework for distinguishing between 'small' and 'large' sets. A subset of a topological space $(\mathcal{T}, \tau)$ is called *nowhere dense* if, and only if the interior of the closure of that set is empty in $(\mathcal{T}, \tau)$. Furthermore, a set in $(\mathcal{T}, \tau)$ is called *meager*/of *first Baire category*, if it can be covered by a countable union of nowhere dense sets. A set is of *second Baire category*, if it is not of first Baire category and, finally, a set is called *co-meager*, if it is the complement of a set of first Baire category. Following [2] and sticking to the concept of 'small' and 'large' sets, in a complete metric space, sets of first Baire category are the 'small' sets, sets of second Baire category are referred to as 'not small' and co-meager sets are the 'large' sets. Moreover, given a topological space $(\mathcal{T}, \tau)$, we call an element of a co-meager set a *typical* element and an element of a

meager set an *atypical* element of that space.
In the family $\mathcal{C}^2$ of all bivariate copulas a typical copula is discrete (even mutually completely dependent, see [3, 17]), whereas a typical extreme-value copula has degenerate discrete component (again see [3]). Following along these lines, using the afore-mentioned results on differentiability and mass distributions, as main result on sizes of subclasses of $\mathcal{C}^d_{ar}$ we finally show that a typical d-dimensional Archimedean copula has non-degenerate discrete component, is not absolutely continuous and has full support. As a direct consequence of this result we can provide a new, significantly simplified proof of [5, Theorem 2.5], stating that for dimension $d = 2$ a typical Archimedean copula is strict. For a selection of further contributions viewing copulas from the Baire category perspective we refer to [4, 5, 6, 9] and the references therein.

The rest of this paper is organized as follows: Section 2, containing notations and definitions used throughout this paper, is divided into two parts: the first subsection focuses on general notation used throughout this paper while the second recalls relevant properties of Archimedean copulas and their interplay with Williamson measures. Section 3 studies regularity and mass distributions of Archimedean copulas. After showing how regularity and measure-theoretic properties of the Williamson measure propagate to the corresponding Archimedean copula (and vice versa) we prove that the family of Archimedean copulas whose discrete, absolutely continuous and singular component have full support $\mathbb{I}^d$ is dense in $(\mathcal{C}^d_{ar}, d_\infty)$. After deriving analogous results for the Kendall distribution function in Section 4, in Section 5 we focus on $(d-1)$-st order partial derivatives $\partial_1 ... \partial_{d-1} C$ of Archimedean copulas C and establish denseness of the subclass of all Archimedean copulas exibiting the pathological differentiability behavior sketched at the beginning of the introduction. Finally, Section 6 provides the afore-mentioned category results for subfamilies of $\mathcal{C}^d_{ar}$.
To improve the flow and readability of the paper, the proofs of several technical lemmas are deferred to the appendices.

2. Notation and Preliminaries

2.1. General notation and definitions

Throughout this contribution we will write $\mathbb{I} = [0, 1]$ and let bold symbols denote vectors. In what follows $\mathcal{C}^d$ with $d \geq 2$ denotes the family of all d-dimensional copulas, i.e., the family of distribution functions (restricted to $\mathbb{I}^d$) of random vectors $\mathbf{X} = (X_1, \dots, X_d)$ fulfilling that each X_i is uniformly distributed on $\mathbb{I}$. For a d-dimensional random vector $\mathbf{X}$ we will write $\mathbf{X} \sim C$ for some $C \in \mathcal{C}^d$ if C is the joint distribution function of $\mathbf{X}$ restricted to $\mathbb{I}^d$. For every dimension $d \geq 2$, the minimum copula is defined as $M(\mathbf{x}) := \min_{1 \leq i \leq d} x_i$ for every $\mathbf{x} \in \mathbb{I}^d$ and the product (independence) copula is defined as $\Pi(\mathbf{x}) = \prod_{i=1}^d x_i$ for every $\mathbf{x} \in \mathbb{I}^d$. Given $C \in \mathcal{C}^d$ we will let μ_C denote the corresponding d-stochastic measure, i.e., the probability measure defined by $\mu_C([\mathbf{0}, \mathbf{x}]) := C(\mathbf{x})$ for all $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{I}^d$ with $[\mathbf{0}, \mathbf{x}] := [0, x_1] \times [0, x_2] \times \ldots \times [0, x_d]$ and extended to the Borel σ-field $\mathcal{B}(\mathbb{I}^d)$ on $\mathbb{I}^d$ in the standard measure-theoretic way.
For $\mathbf{X} \sim C$ the Kendall distribution function F^d_K of C is the distribution function of the (univariate)

random variable $C(\mathbf{X})$, i.e., we have

$$F_K^d(t) := \mathbb{P}(C(\mathbf{X}) \leq t)$$

for every $t \in \mathbb{I}$.

Simplifying notation, for every pair $(i, j) \in \{1, ..., d\}^2$ with $i < j$ and $\mathbf{x} \in \mathbb{I}^d$ we will write $\mathbf{x}_{i:j} := (x_i, \dots, x_j)$. Moreover, for $C \in \mathcal{C}^d$ and $1 \leq m \leq d$ the marginal copula $C^{1:m}$ of the first m coordinates of C is defined by $C^{1:m}(x_1, x_2, \dots, x_m) := C(x_1, x_2, \dots, x_m, 1, \dots, 1)$ for all $\mathbf{x} = (x_1, x_2, ..., x_m) \in \mathbb{I}^m$. The uniform metric d_∞ on $\mathcal{C}^d$ is given by

$$d_\infty(A, B) := \sup_{\mathbf{x} \in \mathbb{I}^d} |A(\mathbf{x}) - B(\mathbf{x})|$$

for all $A, B \in \mathcal{C}^d$. It is well known (see, e.g., [8, 22]) that $(\mathcal{C}^d, d_\infty)$ is a compact metric space. For more background on copulas and d-stochastic measures we refer to [8, 22].

Given an arbitrary topological space (S, τ) the Borel σ-field on S will be denoted by $\mathcal{B}(S)$, the family of all probability measures and of all (positive) finite measures on $\mathcal{B}(S)$ by $\mathcal{P}(S)$ and $\mathcal{M}(S)$, respectively. Focusing on Polish spaces S the topology induced by weak convergence of elements in $\mathcal{P}(S)$ will be denoted by τ_w (see, e.g., [1]). For every $\nu \in \mathcal{M}(S)$ the support $\mathrm{supp}(\nu)$ of ν is the complement of the union of all open sets U with the property that $\nu(U) = 0$. We say that ν has full support if $\mathrm{supp}(\nu) = S$ or, equivalentely, if $\nu(U) > 0$ holds for every open set U in S. In the sequel the support of a copula $C \in \mathcal{C}^d$ is (by definition) the support of its corresponding d-stochastic measure μ_C. As a direct consequence, $C \in \mathcal{C}^d$ has full support if, and only if $\mathrm{supp}(\mu_C) = \mathbb{I}^d$ holds.

The Lebesgue measure on $\mathcal{B}(\mathbb{I}^d)$ or $\mathcal{B}(\mathbb{R}^d)$ will be denoted by λ_d, if the dimension is equal to 1 we will drop the index and simply write λ instead of λ_1. The Dirac measure at some point $x \in S$ is denoted by δ_x. Letting (S, d) and (S', d') denote two metric (or, more general, two topological) spaces, $T : S \to S'$ be a Borel-measurable transformation and $\nu \in \mathcal{P}(S)$, then the push-forward (measure) $\nu^T \in \mathcal{P}(S')$ of ν via T is defined by $\nu^T(F) := \nu(T^{-1}(F))$ for all $F \in \mathcal{B}(S')$.

In what follows conditional distributions and Markov kernels (regular conditional distributions) will play an important role. Given some $m \in \{1, \dots, d-1\}$ we call a map $K : \mathbb{R}^m \times \mathcal{B}(\mathbb{R}^{d-m}) \to \mathbb{I}$ an *m-Markov kernel* from $\mathbb{R}^m$ to $\mathbb{R}^{d-m}$, if the function $\mathbf{x} \mapsto K(\mathbf{x}, E)$ is $\mathcal{B}(\mathbb{R}^m)$-$\mathcal{B}(\mathbb{R}^{d-m})$-measurable for every fixed $E \in \mathcal{B}(\mathbb{R}^{d-m})$ and the map $E \mapsto K(\mathbf{x}, E)$ is a probability measure on $\mathcal{B}(\mathbb{R}^{d-m})$ for every $\mathbf{x} \in \mathbb{R}^m$. If for every $\mathbf{x} \in \mathbb{I}^m$ the measure $K(\mathbf{x}, \cdot)$ only fulfills $K(\mathbf{x}, \mathbb{I}^{d-m}) \leq 1$ instead of $K(\mathbf{x}, \mathbb{I}^{d-m}) = 1$ then we call K an m-sub-Markov kernel.

Given a $(d-m)$-dimensional random vector $\mathbf{Y}$ and an m-dimensional random vector $\mathbf{X}$ on a joint probability space $(\Omega, \mathcal{A}, \mathbb{P})$, a Markov kernel K is called a regular conditional distribution of $\mathbf{Y}$ given $\mathbf{X}$, if (and only if) for every set $E \in \mathcal{B}(\mathbb{R}^{d-m})$ the identity

$$K(\mathbf{X}(\omega), E) = \mathbb{E}(\mathbf{1}_E \circ \mathbf{Y} | \mathbf{X})(\omega)$$

holds for $\mathbb{P}$-almost every $\omega \in \Omega$. It is a well-known fact (see [12, 18]) that for each pair $(\mathbf{X}, \mathbf{Y})$ of random vectors such a regular conditional distribution K of $\mathbf{Y}$ given $\mathbf{X}$ exists and that it is unique for $\mathbb{P}^{\mathbf{X}}$-almost every $\mathbf{x} \in \mathbb{R}^m$. For $(\mathbf{X}, \mathbf{Y}) \sim C$ we will let $K_C : \mathbb{I}^m \times \mathcal{B}(\mathbb{I}^{d-m}) \to \mathbb{I}$ denote (a version of) the corresponding conditional distribution of $\mathbf{Y}$ given $\mathbf{X}$; K_C will simply be referred to as (a version of) the *m-Markov kernel* of the copula C.

For every $G \subseteq \mathbb{I}^d$ and $\mathbf{x} \in \mathbb{I}^m$ define the $\mathbf{x}$-section $G_\mathbf{x}$ of G by $G_\mathbf{x} := \{\mathbf{y} \in \mathbb{I}^{d-m} : (\mathbf{x}, \mathbf{y}) \in G\} \in \mathcal{B}(\mathbb{I}^{d-m})$. Applying *disintegration* of μ_C into the marginal $\mu_{C^{1:m}}$ and the m-Markov kernel K_C of C (see [12, Section 5] and [18, Section 8]) the following identity holds for all $G \in \mathcal{B}(\mathbb{I}^d)$:

$$\mu_C(G) = \int_{\mathbb{I}^m} K_C(\mathbf{x}, G_\mathbf{x}) \,\mathrm{d}\mu_{C^{1:m}}(\mathbf{x}). \tag{1}$$

For more background on conditional expectation, conditional distributions and Markov kernels we refer to [12, 18].

Throughout this article a measure $\nu \in \mathcal{M}(S)$ with $S = \mathbb{I}^d$ or $S = [0, \infty)$ is called singular (w.r.t. λ_d or w.r.t. to λ, respectively) if ν does not have any point-masses and if there exists some $G \in \mathcal{B}(S)$ such that $\lambda_d(G) = 0$ and $\nu(G) = \nu(S)$ hold. We will refer to such measures simply as 'singular' (instead of the alternative 'singular without point-masses') for the sake of simplicity, for being in accordance with singular distribution functions, and because of the subsequent simple observation: For arbitrary $C \in \mathcal{C}^d$ the corresponding d-stochastic measure μ_C obviously has no point-masses, i.e., its discrete component in the sense of Lebesgue decomposition is degenerate. Proceeding analogously to [15] and decomposing the $(d-1)$-kernels K_C, however, allows to decompose C into three not necessarily degenerate components: in fact, denoting the absolutely continuous, the discrete and the singular $(d-1)$-sub-kernels of K_C by K_C^{abs}, K_C^{dis}, $K_C^{sing} \colon \mathbb{I}^{d-1} \times \mathcal{B}(\mathbb{I}) \to \mathbb{I}$, respectively, according to [19] we have that

$$K_C(\mathbf{x}, F) = K_C^{abs}(\mathbf{x}, F) + K_C^{dis}(\mathbf{x}, F) + K_C^{sing}(\mathbf{x}, F) \tag{2}$$

holds for every $\mathbf{x} \in \mathbb{I}^{d-1}$ and every $F \in \mathcal{B}(\mathbb{I})$. Using this decomposition, assuming that the copula $C \in \mathcal{C}^d$ is exchangeable (see [8, Definition 1.7.12] for a formal definition), and that the marginal measure $\mu_{C^{1:d-1}}$ is absolutely continuous, disintegration allows us to decompose μ_C into the three measures μ_C^{abs}, μ_C^{dis}, μ_C^{sing}, defined by

$$\begin{aligned}
\mu_C^{abs}(E \times F) &:= \int_E K_C^{abs}(\mathbf{x}, F) \mathrm{d}\mu_{C^{1:d-1}}(\mathbf{x}), \\
\mu_C^{dis}(E \times F) &:= \int_E K_C^{dis}(\mathbf{x}, F) \mathrm{d}\mu_{C^{1:d-1}}(\mathbf{x}), \\
\mu_C^{sing}(E \times F) &:= \int_E K_C^{sing}(\mathbf{x}, F) \mathrm{d}\mu_{C^{1:d-1}}(\mathbf{x}),
\end{aligned} \tag{3}$$

for all $E \in \mathcal{B}(\mathbb{I}^{d-1})$ and $F \in \mathcal{B}(\mathbb{I})$ and extended to full $\mathcal{B}(\mathbb{I}^d)$ in the standard way. To avoid ambiguity with concepts already discussed in the literature we will refer to these three measures as the absolutely continuous, the discrete and the singular (sub-probability) measures induced by the $(d-1)$-kernel K_C and the marginal $C^{1:d-1}$, respectively. Notice that the definition in equation (3) is directly applicable to Archimedean copulas $C \in \mathcal{C}_{ar}^d$, since these copulas are exchangeable and have absolutely continuous $(d-1)$-marginals $\mu_{C^{1:d-1}}$.

Considering a real function u, the left-hand and right-hand derivatives of u (assuming their existence) will be denoted by D^-u and D^+u, respectively; $u^{(m)}$ will denote the m-th derivative of u (wherever it exists). A function $u\colon (a,b) \to \mathbb{R}$, whereby $(a,b) \subseteq \mathbb{R}$ denotes an open, not necessarily finite interval, is called d-monotone with $2 \leq d \in \mathbb{N}$, if (i) it is differentiable up to order $d-2$, if (ii)

$$(-1)^k u^{(k)}(x) \geq 0,$$

holds for all $k \in \{0, ..., d-2\}$ and $x \in (a,b)$, and if (iii) $(-1)^{d-2}u^{(d-2)}$ is non-increasing and convex on (a,b). Finally, a real-valued function u defined on $[0,\infty)$ is called d-monotone, if it is continuous on $[0,\infty)$ and d-monotone on $(0,\infty)$.

2.2. Multivariate Archimedean copulas

In what follows we will consistently use the conventions $\inf \emptyset := \infty$, $\frac{1}{\infty} := 0$, $\frac{1}{0} := \infty$. We call a non-increasing and continuous function $\psi\colon [0,\infty) \to \mathbb{I}$ which fulfills $\psi(0) = 1$, $\lim_{z\to\infty} \psi(z) = 0 =: \psi(\infty)$ and which is strictly decreasing on $[0, \inf\{x\colon \psi(x) = 0\})$ an Archimedean generator; in the sequel we will simply refer to Archimedean generators as generators. The pseudo inverse $\varphi\colon \mathbb{I} \to [0,\infty]$ of a generator ψ is defined by $\varphi(y) := \inf\{z \in [0,\infty]\colon \psi(z) = y\}$ for every $y \in \mathbb{I}$. It is straightforward to see that φ is strictly decreasing on $(0,1]$, right-continuous at 0 and fulfills $\varphi(1) = 0$ (see [15]). A generator ψ (or its pseudo-inverse φ) is called strict if $\varphi(0) = \infty$ (or equivalently if $\psi(z) > 0$ holds for all $z \in [0,\infty)$). A copula $C \in \mathcal{C}^d$ is called Archimedean if there exists an Archimedean generator ψ such that

$$C(\mathbf{x}) := \psi\left(\sum_{i=1}^{d} \varphi(x_i)\right) \tag{4}$$

holds for all $\mathbf{x} \in \mathbb{I}^d$. In order to stress the correspondence between generator ψ and copula C in the sequel we sometimes write C_ψ instead of C. According to [21, Theorem 2.2], $C(\mathbf{x}) := \psi\left(\sum_{i=1}^{d} \varphi(x_i)\right)$ is a d-dimensional Archimedean copula if, and only if ψ is d-monotone on $[0,\infty)$, which, in turn is equivalent to the fact that $(-1)^{d-2}\psi^{(d-2)}$ exists on $(0,\infty)$, is non-negative, non-increasing and convex on $(0,\infty)$, see [21, Proposition 2.3]. Convexity of $(-1)^{d-2}\psi^{(d-2)}$ implies that both $D^-\psi^{(d-2)}(z)$ and $D^+\psi^{(d-2)}(z)$ exist on $(0,\infty)$, that the one-sided derivatives coincide outside a countable set (see [13, Theorem 3.7.4] and [26, Appendix C]) and that $D^-\psi^{(d-2)}(z) = D^+\psi^{(d-2)}(z)$ holds for every continuity point z of $D^-\psi^{(d-2)}$. Furthermore (see [15]), every d-monotone generator ψ fulfills $\psi^{(m)}(\infty) := \lim_{z\to\infty} \psi^{(m)}(z) = 0$ for every $m \in \{0, ..., d-2\}$ as well as $D^-\psi^{(d-2)}(\infty) := \lim_{z\to\infty} D^-\psi^{(d-2)}(z) = 0$. Note that the convexity of ψ implies that its pseudo-inverse φ is convex as well.

In order to have a one-to-one correspondence between generators and Archimedean copulas in what follows we always assume the generator ψ to be normalized in the sense that $\psi(1) = \frac{1}{2}$ (or equivalently $\varphi(\frac{1}{2}) = 1$) holds and denote the family of all normalized d-monotone generators by $\mathbf{\Psi}_d$. The class of all d-dimensional Archimedean copulas will be denoted by $\mathcal{C}^d_{ar}$. Moreover an Archimedean copula $C_\psi \in \mathcal{C}^d_{ar}$ will be called strict (non-strict) if its corresponding generator ψ is strict (non-strict). Throughout this contribution $\mathcal{C}^d_{ar,s}$ and $\mathcal{C}^d_{ar,n}$ will denote the subclasses of all strict and non-strict d-dimensional Archimedean copulas, respectively. According to [21] $C \in \mathcal{C}^d_{ar}$ is absolutely continuous if, and only if $\psi^{(d-1)}$ exists and is absolutely continuous on $(0,\infty)$. Building upon this fact, in the absolutely continuous case (a version of) the density c of C is given by

$$c(\mathbf{x}) = \mathbf{1}_{(0,1)^d}(\mathbf{x}) \prod_{i=1}^{d} \varphi'(x_i) \cdot D^{-}\psi^{(d-1)} \left(\sum_{i=1}^{d} \varphi(x_i) \right) \tag{5}$$

for all $\mathbf{x} \in \mathbb{I}^d$. As established in [21, Proposition 4.1], every at most $(d-1)$-dimensional marginal of a d-dimensional Archimedean copula is absolutely continuous. This fact allows us to work with the definition of the absolutely continuous, the discrete and the singular component of an Archimedean copula $C \in \mathcal{C}^d_{ar}$ according to equation (3); the families of all absolutely continuous, of all discrete and of all singular Archimedean copulas will be denoted by $\mathcal{C}^d_{ar,abs}$, $\mathcal{C}^d_{ar,dis}$ and $\mathcal{C}^d_{ar,sing}$, respectively. Defining the family $\mathcal{P}^d_{nor}$ of all d-normalized probability measures by

$$\mathcal{P}^d_{nor} := \left\{ \gamma \in \mathcal{P}([0,\infty)) : \int_{\mathbb{I}} (1-t)^{d-1} \mathrm{d}\gamma(t) = \frac{1}{2} \right\},$$

it is straightforward to verify that $\mathcal{P}^d_{nor}$ is weakly closed in $\mathcal{P}([0,\infty))$. Considering that $\mathcal{P}([0,\infty))$ is a complete metric space (see [25]), $\mathcal{P}^d_{nor}$ is (as closed subset of a complete metric space) complete as well. Motivated by the results in [21], characterizing Archimedean copulas via so-called Williamson measures we define the family of all normalized d-Williamson measures $\mathcal{P}_{\mathcal{W}_d}$ by

$$\mathcal{P}_{\mathcal{W}_d} := \left\{ \gamma \in \mathcal{P}([0,\infty)) : \gamma(\{0\}) = 0 \text{ and } \int_{\mathbb{I}} (1-t)^{d-1} \mathrm{d}\gamma(t) = \frac{1}{2} \right\}. \tag{6}$$

If the dimension d is clear from the context we simply speak of Williamson measures. Throughout this article we use the convention that $\gamma(\{\infty\}) := 0$. Following [15, 21] there is a one-to-one correspondence between the family of normalized d-Williamson-measures $\mathcal{P}_{\mathcal{W}_d}$ and the family $\mathbf{\Psi}_d$ of all normalized d-monotone Archimedean generators. In fact, the mapping $\mathcal{W}_d$ (usually referred to as the Williamson d-transform), defined by

$$\psi(z) := (\mathcal{W}_d\gamma)(z) := \int_{[0,\infty)} (1-tz)_+^{d-1} \mathrm{d}\gamma(t), \quad z \in [0,\infty), \tag{7}$$

maps $(\mathcal{P}_{\mathcal{W}_d}, \tau_w)$ to $(\Psi_d, \|\cdot\|_\infty)$.

Remark 2.1. Let $(U_1, U_2, \ldots, U_d) \sim C \in \mathcal{C}^d_{ar}$ and $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ denote the corresponding Williamson measure. According to [21, Theorem 3.1], the transformed vector

$$\mathbf{Z} = (\varphi(U_1), \varphi(U_2), \ldots, \varphi(U_d))$$

follows an ℓ_1-norm symmetric distribution. Specifically, $\mathbf{Z}$ has the same distribution as $R\mathbf{S}_d$, where R is a nonnegative random variable and $\mathbf{S}_d$ is a random vector uniformly distributed on the unit simplex

$$\mathcal{S}_d = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_1 = 1\},$$

with $\|\cdot\|_1$ denoting the usual ℓ_1-norm. Moreover, the random variable R and the random vector $\mathbf{S}_d$ are independent.
Letting F_R denote the distribution function of R and μ_{F_R} the associated probability measure, in view of [21, Definition 3.2], it follows that the Williamson measure γ corresponding to C is given by $\mu_{F_R}^T$, where the transformation $T : [0, \infty] \to [0, \infty]$ is given by $T(s) := \frac{1}{s}$.
We work with the Williamson measure γ rather than the distribution function F_R of the radial component R to assure consistency with the results presented in [15]. In particular, our approach relies on the characterization of d-monotone functions as given in [28, Theorem 1.11], which in turn originates from the foundational work [30].

Building upon the afore-mentioned one-to-one correspondence between normalized d-monotone Archimedean generators $\psi \in \mathbf{\Psi}_d$ and d-dimensional Archimedean copulas $C \in \mathcal{C}_{ar}^d$ the map ξ_d implicitly defined by equation (4) maps $(\Psi_d, \|\cdot\|_\infty)$ to $(\mathcal{C}_{ar}, d_\infty)$. It is straightforward to show that both maps $\mathcal{W}_d$ and ξ_d are homeomorphisms.

Lemma 2.2. *The maps $\mathcal{W}_d$, ξ_d and $\xi_d \circ \mathcal{W}_d$ are homeomorphisms.*

Proof. According to [15, Theorem 5.1.] the map $\mathcal{W}_d$ is a bijection, hence applying [15, Theorem 5.9.] yields that both $\mathcal{W}_d$ and its inverse are continuous. Working with normalized generators and again following [15] implies that the map ξ_d is bijective. Using [15, Theorem 4.1] yields continuity of ξ_d and its inverse ξ_d^{-1}. Since compositions of homeomorphisms are homeomorphisms, this completes the proof. □

It is straightforward to derive from equation (7) that for every $r \in (0, \infty)$ we have $\gamma([0, r)) = 0$ if, and only if $\psi(\frac{1}{r}) = 0$. Moreover, from [15, Lemma 5.5] stating that $C \in \mathcal{C}_{ar}^d$ is strict if, and only if the support of the corresponding Williamson measure $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ contains the point 0, we may define the family of all Williamson measures associated with strict d-dimensional Archimedean copulas by

$$\mathcal{P}_{\mathcal{W}_d}^s := \{\gamma \in \mathcal{P}_{\mathcal{W}_d} \colon \gamma((0, r)) > 0 \text{ for every } r > 0\}$$

and the family of all Williamson measures associated with non-strict Archimedean copulas by

$$\mathcal{P}_{\mathcal{W}_d}^n := \{\gamma \in \mathcal{P}_{\mathcal{W}_d} \colon \exists r > 0 \text{ with } \gamma((0, r)) = 0\}.$$

Finally, $\mathcal{P}_{\mathcal{W}_d}^{fs}$ will denote the set of all normalized d-Williamson measures having full support $[0, \infty)$, the families of all (purely) absolutely continuous, discrete and singular normalized d-Williamson measures will be denoted by $\mathcal{P}_{\mathcal{W}_d}^{abs}$, $\mathcal{P}_{\mathcal{W}_d}^{dis}$ and $\mathcal{P}_{\mathcal{W}_d}^{sing}$, respectively.

Throughout this article we will frequently work with the level sets of Archimedean copulas C, i.e., the sets of all points $\mathbf{x} \in \mathbb{I}^d$ with $C(\mathbf{x}) = t$ for some predefined $t \in \mathbb{I}$. For fixed $t \in (0, 1]$ the t-level set $L_t^{1:d} := L_t$ of $C \in \mathcal{C}_{ar}^d$ is given by

$$\begin{aligned} L_t &:= \{(\mathbf{x}, y) \in \mathbb{I}^{d-1} \times \mathbb{I} \colon C(\mathbf{x}, y) = t\} \\ &= \left\{ (\mathbf{x}, y) \in \mathbb{I}^{d-1} \times \mathbb{I} \colon \sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y) = \varphi(t) \right\} \end{aligned} \tag{8}$$

and for $t = 0$ by

$$\begin{aligned} L_0 &:= \{(\mathbf{x}, y) \in \mathbb{I}^{d-1} \times \mathbb{I} \colon C(\mathbf{x}, y) = 0\} \\ &= \left\{(\mathbf{x}, y) \in \mathbb{I}^{d-1} \times \mathbb{I} \colon \sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y) \geq \varphi(0)\right\}. \end{aligned} \tag{9}$$

The t-level sets of the marginal copula $C^{1:d-1}$ are defined analogously and denoted by $L_t^{1:d-1}$ for every $t \in \mathbb{I}$. Moreover, the function $f^0 \colon \mathbb{I}^{d-1} \to \mathbb{I}$, whose graph coincides with the upper boundary of L_0, is defined by

$$f^0(\mathbf{x}) := \begin{cases} 1, & \text{if } \mathbf{x} \in L_0^{1:d-1}, \\ \psi\left(\varphi(0) - \sum_{i=1}^{d-1} \varphi(x_i)\right), & \text{if } \mathbf{x} \notin L_0^{1:d-1}, \end{cases} \tag{10}$$

whereby we set $\psi(u) = 1$ for all $u < 0$. Notice that equation (10) holds both, in the strict and the non-strict setting: In the strict case we have $\varphi(0) = \infty$, so for $\mathbf{x} \notin L_0^{1:d-1}$ we have $\sum_{i=1}^{d-1} \varphi(x_i) \in [0, \infty)$, implying $\varphi(0) - \sum_{i=1}^{d-1} \varphi(x_i) = \infty$ and $f^0(\mathbf{x}) = \psi(\infty) = 0$; hence in the strict case we have $f^0(\mathbf{x}) \in \{0, 1\}$ for every $\mathbf{x} \in \mathbb{I}^{d-1}$. Contrary to that, in the non-strict case f^0 also attains values in $(0, 1)$.
For $t \in (0, 1]$ we will also use the upper t-cut $[C]_t$ of C, defined by

$$[C]_t := \{\mathbf{x} \in \mathbb{I}^d \colon C(\mathbf{x}) \geq t\}.$$

The upper t-cuts of marginal copulas are defined analogously. The so-called t-level function $f^t \colon [C^{1:d-1}]_t \to \mathbb{I}$ is given by

$$f^t(\mathbf{x}) := \psi\left(\varphi(t) - \sum_{i=1}^{d-1} \varphi(x_i)\right) \tag{11}$$

for every $\mathbf{x} \in [C^{1:d-1}]_t$. It is straightforward to verify that for $t \in (0, 1]$ the graph of f^t coincides with L_t. Moreover, as proved in the next section, the discrete component of an Archimedean copula (if any) is always concentrated on the graphs of some f^t with $t \in [0, 1]$. According to [15, 21] the mass of the t-level sets can be explicitly calculated - in fact, for every $t \in (0, 1)$

$$\mu_C(L_t) = \frac{(-\varphi(t))^{d-1}}{(d-1)!} (D^- \psi^{(d-2)}(\varphi(t)) - D^- \psi^{(d-2)}(\varphi(t)+)) \tag{12}$$

holds and for $t = 0$ we have

$$\mu_C(L_0) = \frac{(-\varphi(0))^{d-1}}{(d-1)!} D^- \psi^{(d-2)}(\varphi(0)) \tag{13}$$

in the non-strict case as well as $\mu_C(L_0) = 0$ in the strict case. For $t = 1$ we obviously have $L_1 = \{(1, \dots, 1)\}$, implying $\mu_C(L_1) = 0$.
Given $0 < s < t \leq 1$, define the set

$$L_{[s,t]} := \left\{(\mathbf{x}, y) \in \mathbb{I}^{d-1} \times \mathbb{I} \colon C(\mathbf{x}, y) \in [s, t]\right\}.$$

Extending the definition of f^s to full $\mathbb{I}^{d-1}$ by setting $f^s(\mathbf{x}) = 1$ for $\mathbf{x} \in \mathbb{I}^{d-1} \setminus [C^{1:d-1}]_s$ obviously the set $L_{[s,t]}$ may also be expressed as

$$L_{[s,t]} = \Big\{(\mathbf{x}, y) \in \mathbb{I}^{d-1} \times \mathbb{I} : f^s(\mathbf{x}) \leq y \leq f^t(\mathbf{x})\Big\}.$$

Considering a d-dimensional Archimedean copula C with generator ψ, then according to [15, 21] the Kendall distribution function F_K^d of C can explicitly be calculated and is given by

$$F_K^d(t) = D^-\psi^{(d-2)}(\varphi(t)) \cdot \frac{(-1)^{d-1}}{(d-1)!}\varphi(t)^{d-1} + \sum_{k=0}^{d-2} \psi^{(k)}(\varphi(t))\frac{(-1)^k}{k!}\varphi(t)^k$$

for every $t \in (0, 1]$ and by

$$F_K^d(0) = \mu_C(L_0) = \frac{(-\varphi(0))^{d-1}}{(d-1)!} D^-\psi^{(d-2)}(\varphi(0))$$

for $t = 0$. Throughout this contribution we denote the probability measure corresponding to the Kendall distribution function F_K^d by $\kappa_{F_K^d}$. As proved in [15, Theorem 5.6], the Kendall distribution function F_K^d and the mass of the t-level sets L_t can be represented directly via the Williamson measure. The following result is a compressed version of [15, Theorem 5.6] using the conventions mentioned at the beginning of Section 2.2. It also follows directly from [21, Proposition 4.5].

Theorem 2.3. *Let C be a d-dimensional Archimedean copula with generator ψ and Williamson measure γ. Then (in the strict and the non-strict case) we have that*

$$\mu_C(L_t) = \gamma\left(\left\{\frac{1}{\varphi(t)}\right\}\right) \tag{14}$$

holds for every $t \in \mathbb{I}$. Moreover, the Kendall distribution function F_K^d of C fulfills

$$F_K^d(t) = \gamma\left(\left[0, \frac{1}{\varphi(t)}\right]\right) \tag{15}$$

for every $t \in \mathbb{I}$.

We conclude this section with an explicit expression for the $(d-1)$-Markov kernel of Archimedean copulas as derived in [15, Theorem 3.1]; this expression will be key in the next sections. Suppose that $C \in \mathcal{C}_{ar}^d$ and let ψ and φ be its associated generator and pseudo-inverse, respectively. Then (a version of) the Markov-kernel K_C of C is given by

$$K_C(\mathbf{x}, [0, y]) := \begin{cases} 1, & M(\mathbf{x}) = 1 \text{ or } \mathbf{x} \in L_0^{1:d-1} \\ 0, & M(\mathbf{x}) < 1, \mathbf{x} \notin L_0^{1:d-1}, y < f^0(\mathbf{x}) \\ \frac{D^-\psi^{(d-2)}\left(\sum_{i=1}^{d-1}\varphi(x_i)+\varphi(y)\right)}{D^-\psi^{(d-2)}\left(\sum_{i=1}^{d-1}\varphi(x_i)\right)}, & M(\mathbf{x}) < 1, \mathbf{x} \notin L_0^{1:d-1}, y \geq f^0(\mathbf{x}), \end{cases} \tag{16}$$

for every $\mathbf{x} \in \mathbb{I}^{d-1}$ and every $y \in \mathbb{I}$, see [15, Theorem 3.1]. Notice that (again see [15]) in the first line of equation (16) we could replace the constant 1 by $F(y)$ for any univariate distribution function F since $L_0^{1:d-1}$ fulfills $\mu_{C^{1:d-1}}(L_0^{1:d-1}) = 0$ and the $(d-1)$-kernel K_C is only unique outside a set of $\mu_{C^{1:d-1}}$-measure 0.

2.3. Summary of Notation

Table 1: Summary of the main notation used throughout the paper

Notation	Description
$\mathbb{I}$	unit interval $[0,1]$
d	dimension
$\mathcal{C}^d$	family of all d-dimensional copulas
M	Fréchet–Hoeffding upper bound (minimum copula)
Π	independence copula
μ_C	d-stochastic measure associated with a copula $C \in \mathcal{C}^d$
F_K^d	Kendall distribution function of a copula $C \in \mathcal{C}^d$
$C^{1:m}$	marginal of the first m coordinates of a copula $C \in \mathcal{C}^d$
d_∞	uniform (supremum) metric
$\mathcal{B}(S)$	Borel σ-field on a topological space (S,τ)
$\mathcal{P}(S)$	set of all probability measures on S
$\mathrm{supp}(\nu)$	support of a measure ν
λ_d, λ	d-dimensional and one-dimensional Lebesgue measure, respectively
δ_x	Dirac measure at a point x
ν^T	push-forward measure of a measure ν under a transformation T
K_C	Markov kernel associated with a copula $C \in \mathcal{C}^d$
$G_{\mathbf{x}}$	$\mathbf{x}$-section of a set G
μ_C^{abs}, μ_C^{dis}, μ_C^{sing}	absolutely continuous, discrete, and singular components of μ_C
$u^{(m)}$	m-th derivative of a function u
$D^{\pm}u$	right- and left-hand derivatives of a function u
ψ	generator of an Archimedean copula
φ	pseudo-inverse of the generator ψ
Ψ_d	family of all normalized d-monotone generators
$\mathcal{C}_{ar}^d$	family of all d-dimensional Archimedean copulas
$\mathcal{C}_{ar,s}^d$, $\mathcal{C}_{ar,n}^d$	families of strict and non-strict Archimedean copulas
$\mathcal{C}_{ar,abs}^d$, $\mathcal{C}_{ar,dis}^d$, $\mathcal{C}_{ar,sing}^d$	families of absolutely continuous, discrete, and singular Archimedean copulas
$\mathcal{P}_{nor}^d$	family of d-normalized probability measures
γ	Williamson measure

Continued on next page

Table 1 – continued from previous page

Notation	Description
$\mathcal{P}_{\mathcal{W}_d}$	family of all Williamson measures
$\mathcal{P}^s_{\mathcal{W}_d}$, $\mathcal{P}^n_{\mathcal{W}_d}$	families of strict and non-strict Williamson measures
$\mathcal{P}^{fs}_{\mathcal{W}_d}$	family of Williamson measures with full support
$\mathcal{P}^{abs}_{\mathcal{W}_d}$, $\mathcal{P}^{dis}_{\mathcal{W}_d}$, $\mathcal{P}^{sing}_{\mathcal{W}_d}$	families of absolutely continuous, discrete, and singular Williamson measures
L_t	t-level set
f^t	function parameterizing the level set L_t
$[C]_t$	upper t-cut of a copula C
$\lVert\cdot\rVert_\infty$	uniform (supremum) norm

3. Regularity and mass distributions of multivariate Archimedean copulas

Looking at the Williamson d-transform (7) not surprisingly $D^-\psi^{(d-2)}$ can also be directly expressed in terms of the corresponding Williamson measure. In fact, following [15, Lemma 5.4] for every d-dimensional Archimedean copula C with generator ψ and (normalized) Williamson measure γ the subsequent lemma holds:

Lemma 3.1. *Let $C \in \mathcal{C}^d_{ar}$ be an Archimedean copula and ψ and $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ be its corresponding generator and Williamson measure, respectively. Then*

$$0 \geq G(z) := (-1)^{d-2} D^-\psi^{(d-2)}(z) = -(d-1)! \int_{(0,\frac{1}{z}]} t^{d-1} \mathrm{d}\gamma(t) \tag{17}$$

holds for every $z > 0$.

Building upon the previous lemma and fixing $\gamma \in \mathcal{P}_{\mathcal{W}_d}$, for the case $M(\mathbf{x}) < 1$, $\mathbf{x} \notin L_0^{1:d-1}$ and $f^0(\mathbf{x}) \leq y$ the Markov kernel K_C of $C \in \mathcal{C}^d_{ar}$ according to equation (16) can be expressed as

$$K_C(\mathbf{x}, [0, y]) = \frac{G(\sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y))}{G(\sum_{i=1}^{d-1} \varphi(x_i))} = \frac{\int_{I_y} t^{d-1} \mathrm{d}\gamma(t)}{\int_{I_1} t^{d-1} \mathrm{d}\gamma(t)}, \tag{18}$$

with the notation $I_y := \left(0, \frac{1}{\sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y)}\right]$ for every $y \in \mathbb{I}$. The last expression in equation (18) is convenient insofar that (as proved in this section) it insinuates that the regularity of the measure γ directly propagates to the Markov kernel K_C. Considering the Lebesgue decomposition $\gamma = \gamma^{abs} +$

$\gamma^{dis} + \gamma^{sing}$ of $\gamma \in \mathcal{P}_{\mathcal{W}_d}$, equation (18) can be written as

$$\begin{aligned} K_C(\mathbf{x}, [0, y]) &= \frac{\int_{I_y} t^{d-1} \mathrm{d}\gamma(t)}{\int_{I_1} t^{d-1} \mathrm{d}\gamma(t)} \\ &= \underbrace{\frac{\int_{I_y} t^{d-1} \mathrm{d}\gamma^{abs}(t)}{\int_{I_1} t^{d-1} \mathrm{d}\gamma(t)}}_{=: H_{\mathbf{x}}^{abs}(y)} + \underbrace{\frac{\int_{I_y} t^{d-1} \mathrm{d}\gamma^{dis}(t)}{\int_{I_1} t^{d-1} \mathrm{d}\gamma(t)}}_{=: H_{\mathbf{x}}^{dis}(y)} + \underbrace{\frac{\int_{I_y} t^{d-1} \mathrm{d}\gamma^{sing}(t)}{\int_{I_1} t^{d-1} \mathrm{d}\gamma(t)}}_{=: H_{\mathbf{x}}^{sing}(y)}. \end{aligned} \tag{19}$$

Notice that the definitions for $H_{\mathbf{x}}^{abs}, H_{\mathbf{x}}^{dis}, H_{\mathbf{x}}^{sing}$ in equation (19) are merely definitions, it is a priori not clear that the chosen notation is meaningful, i.e., that the function $H_{\mathbf{x}}^{abs}$ (if being non-degenerate) is absolutely continuous, that $H_{\mathbf{x}}^{dis}$ (if being non-degenerate) is discrete and that $H_{\mathbf{x}}^{sing}$ (if being non-degenerate) is singular. These points, however, will be established in the proofs of Lemma 3.9, Lemma 3.10 and in Remark 3.11.

It has already been established (see [15, Theorem 5.12]) that absolute continuity, discreteness and singularity of the Williamson-measure $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ propagates to the corresponding Archimedean copula $C \in \mathcal{C}_{ar}^d$. The goal of this section is to extend this result to the following equivalence (see Theorem 3.12): the absolutely continuous/discrete/singular component of the Williamson measure γ is non-degenerate if, and only if the absolutely continuous/discrete/singular component of the corresponding Archimedean copula C is non-degenerate. Having this stronger result provides a simple proof of the reverse implication in [15, Theorem 5.12].

We start with some preliminary observations which will be key for proving the afore-mentioned equivalence. The following Theorem is a consequence of Theorem 2.3.

Theorem 3.2. *Let C be a d-dimensional Archimedean copula and $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ its corresponding Williamson measure. Moreover, let $s_1, s_2 \in \mathbb{I}$, with $s_1 \leq s_2$. Then the following assertion holds:*

$$\mu_C(L_{[s_1,s_2]}) = \gamma\left(\left[\frac{1}{\varphi(s_1)}, \frac{1}{\varphi(s_2)}\right]\right). \tag{20}$$

Proof. Setting $M_t := \{(\mathbf{x}, y) \in \mathbb{I}^d \colon C(\mathbf{x}, y) \leq t\}$ for every $t \in \mathbb{I}$ we obviously have $F_K^d(t) = \mu_C(M_t)$ for every $t \in \mathbb{I}$. For $s_1 \leq s_2$ considering

$$L_{[s_1,s_2]} = M_{s_2} \cap [C]_{s_1} = M_{s_2} \setminus [C]_{s_1}^c, \quad [C]_{s_1}^c \subseteq M_{s_2},$$

using Theorem 2.3 directly yields

$$\begin{aligned} \mu_C(L_{[s_1,s_2]}) &= \mu_C(M_{s_2} \setminus [C]_{s_1}^c) = \mu_C(M_{s_2}) - \mu_C([C]_{s_1}^c) = F_K^d(s_2) - F_K^d(s_1-) \\ &= \gamma\left(\left[\frac{1}{\varphi(s_1)}, \frac{1}{\varphi(s_2)}\right]\right). \end{aligned}$$

□

The next corollary states a close interrelation between the support of the Williamson measure $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ and the support of the corresponding Archimedean copula $C \in \mathcal{C}^d_{ar}$.

Corollary 3.3. *Suppose that $C \in \mathcal{C}^d_{ar}$, let $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ be its associated Williamson measure, ψ its generator, and φ the pseudo-inverse of ψ. Then for $[a,b] \subseteq [\frac{1}{\varphi(0)}, \infty)$ with $a < b$ the following two conditions are equivalent:*

1. $\gamma([a,b]) = 0$
2. $\mu_C(L_{[\psi(\frac{1}{a}),\psi(\frac{1}{b})]}) = 0$

Proof. Immediate consequence of Theorem 3.2. □

Remark 3.4. *The condition $[a,b] \subseteq [\frac{1}{\varphi(0)}, \infty)$ can not be avoided. In fact, in the non-strict setting for the situation $b < \frac{1}{\varphi(0)}$ it follows that $\gamma([0,b]) = 0$, implying $\gamma([a,b]) = 0$. Nevertheless in this case we have $\psi(\frac{1}{a}) = \psi(\frac{1}{b}) = 0$ but $\mu_C(L_{[0,0]}) > 0$ may hold.*

The subsequent example illustrates Corollary 3.3.

Example 3.5. Consider the distribution function

$$F_\gamma(z) := \begin{cases} 0, & \text{if } z \in [0, \frac{1}{4}), \\ \frac{2}{3}, & \text{if } z \in [\frac{1}{4}, 1), \\ \frac{1}{9}z + \frac{5}{9}, & \text{if } z \in [1, 2), \\ \frac{7}{9}, & \text{if } z \in [2, 3), \\ \frac{1}{9}z + \frac{4}{9}, & \text{if } z \in [3, 4), \\ 1, & \text{if } z \geq 4. \end{cases}$$

It is straightforward to verify that F_γ is the distribution function of a unique Williamson measure $\gamma \in \mathcal{P}_{\mathcal{W}_2}$, whose induced non-strict generator ψ is given by

$$\psi(z) := \begin{cases} 1 - \frac{7z}{6}, & \text{if } z \in [0, \frac{1}{4}], \\ \frac{3z^2+8z+1}{18z}, & \text{if } z \in (\frac{1}{4}, \frac{1}{3}], \\ \frac{1}{9}(7 - 3z), & \text{if } z \in (\frac{1}{3}, \frac{1}{2}], \\ -\frac{2z^2-10z-1}{18z}, & \text{if } z \in (\frac{1}{2}, 1], \\ \frac{2}{3}(1 - \frac{z}{4}), & \text{if } z \in (1, 4], \\ 0, & \text{if } z > 4. \end{cases}$$

The distribution function F_γ and the generator ψ are depicted in Figure 1. Considering the support of the corresponding Archimedean copula C_ψ, we obtain that

$$\operatorname{supp}(\mu_{C_\psi}) = \Gamma(f^0) \cup L_{[\frac{1}{2},\frac{11}{18}]} \cup L_{[\frac{2}{3},\frac{17}{24}]}.$$

A sample of C_ψ is depicted in Figure 2.

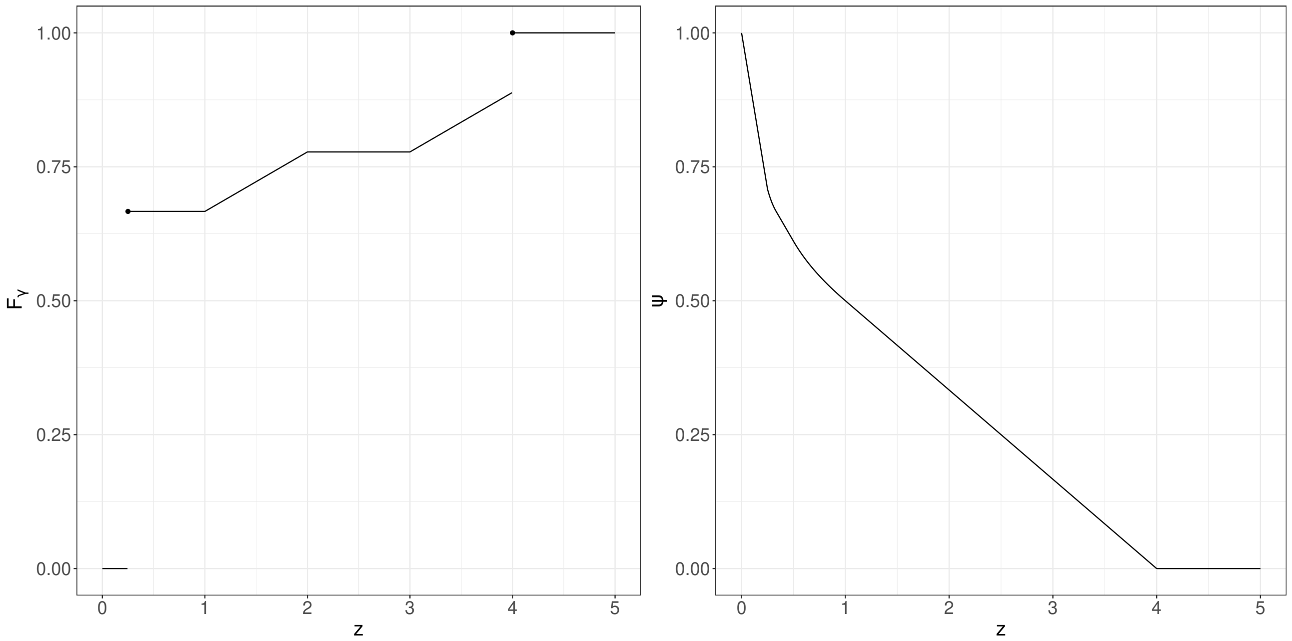

Figure 1: The distribution function F_γ (left panel) and the associated generator ψ (right panel) considered in Example 3.5.

According to [15, Lemma 5.5] strictness of an Archimedean copula C can be characterized in terms of the corresponding Williamson measure. Corollary 3.3 allows to extend this result and show that an Archimedean copula $C \in \mathcal{C}_{ar}^d$ has full support if, and only if its corresponding Williamson measure γ has full support.

Theorem 3.6. *Let $C \in \mathcal{C}_{ar}^d$ and $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ be its corresponding Williamson measure. Then γ has full support $[0, \infty)$ if, and only if C has full support $\mathbb{I}^d$.*

Proof. Suppose that $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ has full support. Then its corresponding distribution function F_γ is strictly increasing, so according to [15, Lemma 5.5] the corresponding Archimedean copula $C \in \mathcal{C}_{ar}^d$ is strict. Fix $\mathbf{x} \notin L_0^{1:d-1}$ with $M(\mathbf{x}) < 1$ and suppose that $y_1, y_2 \in \mathbb{I}$ fulfill $f^0(\mathbf{x}) = 0 \leq y_1 < y_2 \leq 1$. Using the fact that γ has full support we have $\gamma(I_{y_2} \setminus I_{y_1}) > 0$, which implies

$$\int_{I_{y_1}} t^{d-1} \mathrm{d}\gamma(t) < \int_{I_{y_2}} t^{d-1} \mathrm{d}\gamma(t).$$

Having this, using equation (18) $K_C(\mathbf{x}, (y_1, y_2]) > 0$ follows and we have shown that the conditional distribution function $y \mapsto K_C(\mathbf{x}, [0, y])$ is strictly increasing. Considering $\mu_{C^{1:d-1}}((L_0^{1:d-1})^c \cap M^{-1}([0, 1))) = 1$ it follows that for $\mu_{C^{1:d-1}}$-almost every $\mathbf{x} \in \mathbb{I}^{d-1}$ the conditional distribution function $y \mapsto K_C(\mathbf{x}, [0, y])$ is strictly increasing. Using the fact that γ has full support and applying equation

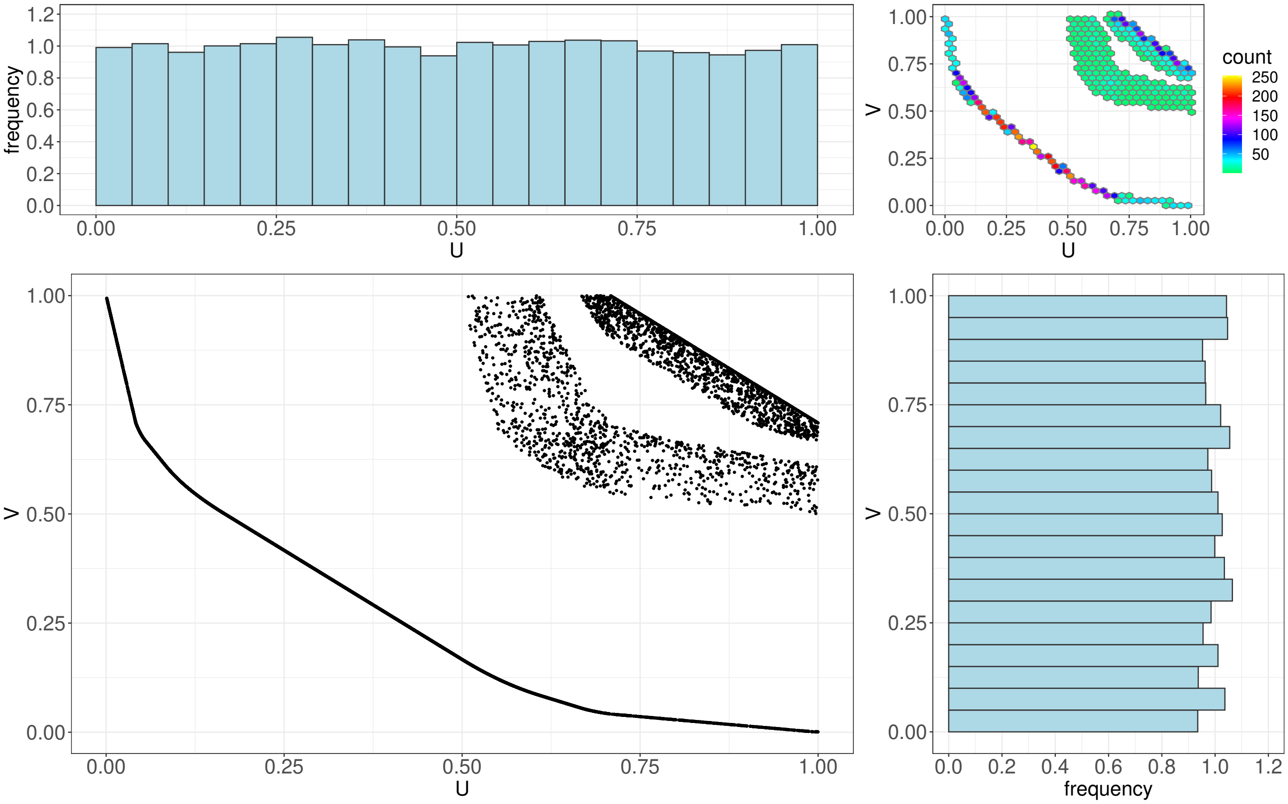

Figure 2: Sample of size 10.000 of the Archimedean copula C_ψ with ψ being the generator from Example 3.5, its histogram and the two marginal histograms; sample generated via conditional inverse sampling.

(17) yields that the density $c^{1:d-1}$ of $\mu_{C^{1:d-1}}$ fulfills

$$c^{1:d-1}(\mathbf{s}) = \prod_{i=1}^{d-1} \varphi'(s_i) D^{-}\psi^{(d-2)} \left(\sum_{i=1}^{d-1} \varphi(s_i) \right) > 0$$

for every $\mathbf{s} \in (0,1)^{d-1}$. Let $R = (\underline{x}_1, \overline{x}_1) \times (\underline{x}_2, \overline{x}_2) \times \cdots \times (\underline{x}_{d-1}, \overline{x}_{d-1}) \times (\underline{y}, \overline{y}) \subset \mathbb{I}^d$ denote an arbitrary open rectangle with $\lambda_d(R) > 0$. Writing $R^{1:d-1} := (\underline{x}_1, \overline{x}_1) \times (\underline{x}_2, \overline{x}_2) \times \cdots \times (\underline{x}_{d-1}, \overline{x}_{d-1})$ applying disintegration yields

$$\begin{aligned} \mu_C(R) &= \int_{R^{1:d-1}} \underbrace{K_C(\mathbf{x}, (\underline{y}, \overline{y}))}_{>0} \, \mathrm{d}\mu_{C^{1:d-1}}(\mathbf{x}) \\ &= \int_{R^{1:d-1}} \underbrace{K_C(\mathbf{x}, (\underline{y}, \overline{y})) c^{1:d-1}(\mathbf{x})}_{>0} \, \mathrm{d}\lambda_{d-1}(\mathbf{x}) > 0. \end{aligned}$$

In other words: Every open rectangle with positive volume has positive mass. This shows $\operatorname{supp}(\mu_C) = \mathbb{I}^d$ and completes the proof of the first implication.
Considering the reverse direction, suppose that γ would not have full support. Then we could find some $a, b \in [0, \infty)$ with $a < b$ such that $\gamma([a, b]) = 0$. If $[a, b] \subseteq [\frac{1}{\varphi(0)}, \infty)$, Corollary 3.3 directly yields a contradiction. If $a < \frac{1}{\varphi(0)}$ then φ has to be non-strict, implying that L_0 has non-empty interior, so $\operatorname{supp}(\mu_C) \neq \mathbb{I}^d$. □

As next step we prove that the discrete component (if any) of an Archimedean copula C is always concentrated on the graph of the t-level set functions f^t for some $t \in [0, 1)$.

Lemma 3.7. *Let $C \in \mathcal{C}^d_{ar}$, ψ be its generator, $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ its corresponding Williamson measure and consider $t_0 \in (0, 1)$ in case that ψ is strict and $t_0 \in [0, 1)$ in case that ψ is non-strict. Then the following assertions are equivalent:*

(i) $\gamma\left(\left\{\frac{1}{\varphi(t_0)}\right\}\right) > 0$,

(ii) $\varphi(t_0)$ *is a point of discontinuity of* $D^-\psi^{(d-2)}$,

(iii) There exists some set $\Lambda \in \mathcal{B}(\mathbb{I}^{d-1})$ *with* $\mu_{C^{1:d-1}}(\Lambda) > 0$ *such that for all* $\mathbf{x} \in \Lambda$ $K_C(\mathbf{x}, \{f^{t_0}(\mathbf{x})\}) > 0$ *holds,*

(iv) $\mu_C(L_{t_0}) > 0$.

The previous lemma has the following direct corollary.

Corollary 3.8. *Let $C \in \mathcal{C}^d_{ar}$ and $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ be the corresponding Williamson measure. Then γ has a point mass if, and only if $\mu_C^{dis}(\mathbb{I}^d) > 0$.*

As next step we tackle the one-sided versions of the previous corollary for the absolutely continuous and the singular components of the Williamson measure.

Lemma 3.9. *Let $C \in \mathcal{C}^d_{ar}$ and $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ be the corresponding Williamson measure. Then $\gamma^{abs}([0, \infty)) > 0$ implies $\mu_C^{abs}(\mathbb{I}^d) > 0$.*

As a next step, we prove the analogous assertion for the singular component of the measure $\gamma \in \mathcal{P}_{\mathcal{W}_d}$.

Lemma 3.10. *Let $C \in \mathcal{C}^d_{ar}$ and $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ be the corresponding Williamson measure. Then $\gamma^{sing}([0, \infty)) > 0$ implies $\mu_C^{sing}(\mathbb{I}^d) > 0$.*

Remark 3.11. Notice that in the proofs of Lemma 3.9 and Lemma 3.10 the construction via z_0 and Λ was key for showing that $H_{\mathbf{x}}^{abs}$ and $H_{\mathbf{x}}^{sing}$ are non-degenerate, but not for showing absolute continuity and singularity. The proofs showed that for $\mathbf{x} \in \Lambda$ the functions $H_{\mathbf{x}}^{abs}$ and $H_{\mathbf{x}}^{sing}$ are purely absolutely continuous and singular, respectively. For the discrete case Corollary 3.8 states that $\gamma^{dis}([0, \infty)) > 0$ implies $\mu_C^{abs}(\mathbb{I}^d) > 0$ but it was not explicitly shown that $H_{\mathbf{x}}^{dis}$ is purely discrete. This observation, however, can be shown analogously to the absolutely continuous and singular case. In fact, defining the set Λ in the same manner as in the proof of Lemma 3.9 only replacing γ^{abs} by γ^{dis} in the definition of z_0 we again have $\mu_{C^{1:d-1}}(\Lambda) > 0$. Moreover, for $\mathbf{x} \in \Lambda$ considering the definition of $H_{\mathbf{x}}^{dis}$ it is straightforward to show that for $\gamma^{dis} = \sum_{j=1}^{\infty} \alpha_j \delta_{z_j}$ the function $H_{\mathbf{x}}^{dis}$ corresponds to the discrete measure with point mass $\frac{\alpha_j z_j^{d-1}}{\int_{I_1} t^{d-1} \mathrm{d}\gamma}$ at the point $\psi(\frac{1}{z_j} - \sum_{i=1}^{d-1} \varphi(x_i))$ for every $j \in \mathbb{N}$.

Combining the afore-mentioned lemmas and corollaries we obtain the main result of this section stating that the absolutely continuous/discrete/singular component of γ is non-degenerate if, and only if the absolutely continuous/discrete/singular component of the corresponding Archimedean copula C is. Theorem 3.12 will also be crucial for proving Theorem 6.5 and Corollary 6.8, the main result of Section 6 on topologically typical Archimedean copulas.

Theorem 3.12. *Let $C \in \mathcal{C}_{ar}^d$ and $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ be its corresponding Williamson measure. Then the following three equivalences hold:*

(i) $\gamma^{dis}([0,\infty)) > 0$ *if, and only if* $\mu_C^{dis}(\mathbb{I}^d) > 0$.

(ii) $\gamma^{abs}([0,\infty)) > 0$ *if, and only if* $\mu_C^{abs}(\mathbb{I}^d) > 0$.

(iii) $\gamma^{sing}([0,\infty)) > 0$ *if, and only if* $\mu_C^{sing}(\mathbb{I}^d) > 0$.

Proof. Equivalence (i) has already been stated in Corollary 3.8. Moreover, for the assertions (ii) and (iii) one implication has already been proved.
Assume now that γ^{abs} is degenerate. Then $\gamma = \gamma^{dis} + \gamma^{sing}$ and at least one of the two components is non-degenerate. Considering $\mathbf{x} \in \mathbb{I}^{d-1} \setminus L_0^{1:d-1}$ with $M(\mathbf{x}) < 1$, proceeding as in the (last parts of the) proofs of Lemma 3.9 and Lemma 3.10, and considering equation (19) once more yields

$$K_C(\mathbf{x}, [0,y]) = H_{\mathbf{x}}^{dis}(y) + H_{\mathbf{x}}^{sing}(y), \quad y \in \mathbb{I},$$

with $H_{\mathbf{x}}^{dis}$ being discrete and $H_{\mathbf{x}}^{sing}$ being singular. In other words, $K_C(\mathbf{x}, [0,y])$ has degenerate absolutely continuous component. The fact that $\mu_C^{abs}(\mathbb{I}^d) = 0$ now follows immediately via disintegration. The remaining implication in assertion (iii) can be proved in the same manner. $\square$

Theorem 3.12 allows to extend [15, Theorem 5.12] to equivalences - the following result holds:

Corollary 3.13. *Let $C \in \mathcal{C}_{ar}^d$ and $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ be the corresponding Williamson measure. Then the following equivalences hold:*

(i) γ *is absolutely continuous if, and only if* $\mu_C^{abs}(\mathbb{I}^d) = 1$.

(ii) γ *is discrete if, and only if* $\mu_C^{dis}(\mathbb{I}^d) = 1$.

(iii) γ *is singular if, and only if* $\mu_C^{sing}(\mathbb{I}^d) = 1$.

Proof. The first assertion has already been established in [21]. Furthermore, sufficiency concerning assertions (ii) and (iii) is part of [15, Theorem 5.12] and it remains to show necessity. Assume that $0 \leq \gamma^{dis}([0,\infty)) < 1$ holds. Then we have $\gamma^{abs}([0,\infty)) > 0$ or $\gamma^{sing}([0,\infty)) > 0$, so applying Theorem 3.12 immediately yields that $0 \leq \mu_C^{dis}(\mathbb{I}^d) < 1$. An analogous argument shows (iii). $\square$

Viewing Theorem 3.6 and again considering the Lebesgue decomposition of the Williamson measure γ we can show the following stronger statement.

Theorem 3.14. *Let $C \in \mathcal{C}_{ar}^d$ and let $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ be its corresponding Williamson measure. Then the following assertions hold*

(i) If γ^{abs} has full support, then μ_C^{abs} has full support.

(ii) If γ^{dis} has full support, then μ_C^{dis} has full support.

(iii) If γ^{sing} has full support, then μ_C^{sing} has full support.

Proof. To prove the first assertion, let $\mathbf{x} \notin L_0^{1:d-1}$ with $M(\mathbf{x}) < 1$ be arbitrary but fixed and suppose that γ^{abs} has full support, which, in particular implies that ψ is strict. It follows that the function $H_{\mathbf{x}}^{abs}$ defined according to equation (19) is strictly increasing and absolutely continuous, so $K_C^{abs}(\mathbf{x},\cdot)$ has support $\mathbb{I}$. Since $\mathbf{x} \notin L_0^{1:d-1}$ with $M(\mathbf{x}) < 1$ was arbitrary, disintegration directly yields that μ_C^{abs} has support $\mathbb{I}^d$.
Considering the second assertion it might be worth noting that discrete Williamson measures $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ with full support exist. For example, considering the probability measure $\beta := \sum_{i=1}^{\infty} 2^{-i}\delta_{q_i}$ with $\{q_1, q_2, \ldots\} := \mathbb{Q} \cap (0,\infty)$ and normalizing it in the sense of [15, Lemma Appendix B.1] yields a discrete Williamson measure $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ with full support. In fact, every discrete Williamson measure γ whose point masses form a dense subset $\mathcal{Q}$ of $[0,\infty)$ has full support. Taking this into consideration, fixing $\mathbf{x} \notin L_0^{1:d-1}$ with $M(\mathbf{x}) < 1$ and assuming that γ^{dis} has full support yields that the function $H_{\mathbf{x}}^{dis}$ (defined as in equation (18)) is strictly increasing and discrete. Applying the same arguments as in the previous case then implies that μ_C^{dis} has full support.
The third assertion can be proved analogously. $\square$

The previous results open the door to deriving some first (to a certain extent surprising) results - we start with denseness of subclasses of Archimedean copulas with full support.

Theorem 3.15. *The following assertions hold:*

(i) The family $\{C \in \mathcal{C}_{ar,abs}^d \colon \operatorname{supp}(\mu_C) = \mathbb{I}^d\}$ is dense in $(\mathcal{C}_{ar}^d, d_\infty)$,

(ii) The family $\{C \in \mathcal{C}_{ar,dis}^d \colon \operatorname{supp}(\mu_C) = \mathbb{I}^d\}$ is dense in $(\mathcal{C}_{ar}^d, d_\infty)$,

(iii) The family $\{C \in \mathcal{C}_{ar,sing}^d \colon \operatorname{supp}(\mu_C) = \mathbb{I}^d\}$ is dense in $(\mathcal{C}_{ar}^d, d_\infty)$.

Proof. We fix $C \in \mathcal{C}_{ar}^d$ and let $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ denote the corresponding Williamson measure. According to Lemma Appendix B.1 there exists a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of absolutely continuous/discrete/singular Williamson measures with full support such that $(\gamma_n)_{n\in\mathbb{N}}$ converges weakly to γ. Applying [15, Theorem 5.9] yields that the sequence $(C_n)_{n\in\mathbb{N}}$ of corresponding Archimedean copulas converges uniformly to C. Since each $\gamma_n \in \mathcal{P}_{\mathcal{W}_d}$ has full support and is absolutely continuous/discrete/singular, applying Theorem 3.6 and Corollary 3.13 shows that each C_n is absolutely continuous/discrete/singular and that $\mathrm{supp}(\mu_C) = \mathbb{I}^d$ holds. This completes the proof since $C \in \mathcal{C}_{ar}^d$ was arbitrary. □

Using a similar idea of proof the following even stronger version of Theorem 3.15, stating that even the family of Archimedean copulas, whose absolutely continuous, discrete and singular components (simultaneously) have full support, is dense, can be shown.

Corollary 3.16. *The family*

$$\{C \in \mathcal{C}_{ar}^d \colon \mathrm{supp}(\mu_C^{abs}) = \mathrm{supp}(\mu_C^{dis}) = \mathrm{supp}(\mu_C^{sing}) = \mathbb{I}^d\}$$

is dense in $(\mathcal{C}_{ar}^d, d_\infty)$.

Proof. Fix $C \in \mathcal{C}_{ar}^d$ and let $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ denote the corresponding Williamson measure. According to Lemma Appendix B.1 there exist sequences $(\gamma_n^{(1)})_{n\in\mathbb{N}}$, $(\gamma_n^{(2)})_{n\in\mathbb{N}}$ and $(\gamma_n^{(3)})_{n\in\mathbb{N}}$ of purely absolutely continuous, discrete and singular Williamson measures having full support which all converge weakly to γ. Considering

$$\gamma_n := \frac{1}{3}\left(\gamma_n^{(1)} + \gamma_n^{(2)} + \gamma_n^{(3)}\right)$$

for every $n \in \mathbb{N}$, using Theorem 3.14, and proceeding as in the proof of Theorem 3.15 the desired result follows. □

4. Regularity of the Kendall distribution function of Archimedean copulas

In this short section we revisit the interplay between the Williamson measure γ and the Kendall distribution function F_K^d of the corresponding copula $C \in \mathcal{C}_{ar}^d$ and show, loosely speaking, that regularity of C (and hence of γ) goes hand in hand with regularity of F_K^d. Note that in the full class of copulas a similar behavior does not hold. In fact, considering $d = 2$ and the minimum copula M, the Kendall distribution function is given by $F_K^2(t) = t$ (see, e.g., [8, Example 3.9.6.]), so M is discrete although its corresponding Kendall distribution function is absolutely continuous.

Theorem 2.3 shows that the Kendall distribution function F_K^d can nicely be represented in terms of γ (and the quasi-inverse φ of the generator ψ). The subsequent lemma provides an expression for γ via F_K^d:

Lemma 4.1. *Consider $C \in \mathcal{C}_{ar}^d$ and let ψ, γ, and F_K^d denote the corresponding generator, Williamson measure and Kendall distribution function, respectively. Then the following identity holds for all $z \in [0, \infty)$:*

$$\gamma([0,z]) = \begin{cases} F_K^d(\psi(\frac{1}{z})), & \text{if } z \in [\frac{1}{\varphi(0)}, \infty) \\ 0, & \text{if } z \in [0, \frac{1}{\varphi(0)}). \end{cases} \tag{21}$$

Moreover, the non-decreasing transformation $h : [0, \infty] \to \mathbb{I}$*, defined by* $h(z) = \psi(\frac{1}{z})$ *fulfills* $h(0) = 0$ *and is non-singular in the sense that for every* $F \in \mathcal{B}([0, \infty))$ *we have that* $\lambda(F) = 0$ *implies* $\lambda^h(F) = 0$*, where* λ^h *denotes the push-forward of* λ *under* h*.*

As a next step we characterize regularity of the Kendall distribution functions in terms of regularity of the Williamson measure.
Consider $(U_1, \dots, U_d) \sim C \in \mathcal{C}^d_{ar}$ with corresponding Williamson measure γ (whose distribution function is denoted by F_γ) and generator ψ. As shown in [21, Section 4.2], there exists a nonnegative random variable R such that $C(U_1, \dots, U_d)$ has the same distribution as $\psi(R)$.
Taking into account the connection between R and the Williamson measure γ mentioned in Section 2, it follows that the transformed variable $T(R)$, with $T(s) = \frac{1}{s}$ for $s \in (0, \infty)$, has distribution function F_γ. Consequently, $C(U_1, \dots, U_d)$ has the same distribution as $(\psi \circ T^{-1})(T(R))$. Using this and exploiting the fact that ψ is non-increasing and d-monotone the following statements are not surprising.

Theorem 4.2. *Let* $C \in \mathcal{C}^d_{ar}$*,* $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ *be the corresponding Williamson measure,* F^d_K *be the Kendall distribution function of* C *and* $\kappa_{F^d_K}$ *the probability measure corresponding to* F^d_K*. Then the following equivalences hold:*

(i) γ *is absolutely continuous if, and only if* $\kappa_{F^d_K}$ *is absolutely continuous.*

(ii) γ *is discrete if, and only if* $\kappa_{F^d_K}$ *is discrete.*

(iii) γ *is singular if, and only if* $\kappa_{F^d_K}$ *is singular.*

Applying the results from the previous section yields the following:

Corollary 4.3. *Let* $C \in \mathcal{C}^d_{ar}$ *and* γ *and* F^d_K *denote the corresponding Williamson measure and Kendall distribution function, respectively. Then the following equivalences hold:*

(i) C *is absolutely continuous if, and only if* $\kappa_{F^d_K}$ *is absolutely continuous.*

(ii) C *is discrete if, and only if* $\kappa_{F^d_K}$ *is discrete.*

(iii) C *is singular if, and only if* $\kappa_{F^d_K}$ *is singular.*

5. Derivatives of multivariate Archimedean copulas

In [3] it was shown that 'derivatives of copulas have to be handled with care', a statement that was underlined by showing the existence of a dense class of bivariate copulas C of the following type: for λ-almost every $x \in (0, 1)$ the partial derivative $\partial_1 C(x, y)$ does not exist on a dense subset of $y \in (0, 1)$.
Motivated by these results in this section we tackle the question if - in the sense of the derivative - analogously pathological copulas also exist in the Archimedean family.
It is well known (and follows directly from the properties of generators) that every $C \in \mathcal{C}^d_{ar}$ is

continuously differentiable up to order $d-2$ (see [21]). Moreover, as mentioned in the introduction, for every fixed $y \in (0,1)$ the partial derivative $\partial_1 ... \partial_{d-1} C(\mathbf{x}, y)$ exists for λ_{d-1}-almost every $\mathbf{x} = (x_1, ..., x_{d-1}) \in \mathbb{I}^{d-1}$.

We will prove in this section that it is not possible to go much further. As main result we will show that $\mathcal{C}^d_{ar}$ contains copulas C of the following pathological type: there exists some set $\Lambda \in \mathcal{B}(\mathbb{I}^{d-1})$ with $\lambda_{d-1}(\Lambda) = 1$ such that for every $\mathbf{x} \in \Lambda$ the $(d-1)$-st order partial derivative $\partial_1 \partial_2 ... \partial_{d-1} C(\mathbf{x}, y)$ does not exist on a dense set of $y \in (0,1)$. The family of all such d-dimensional Archimedean copulas will be denoted by $\mathcal{C}^d_{ar,\mathcal{Q}}$. As in [3] we first study the class $\mathcal{C}^d_{ar,p}$ of all d-dimensional Archimedean copulas fulfilling that for every $\mathbf{x}$ from a set $\Lambda \in \mathcal{B}(\mathbb{I}^{d-1})$ with $\mu_{C^{1:d-1}}(\Lambda) > 0$ there exists some $y := y_{\mathbf{x}} \in (0,1)$ such that the $(d-1)$-st order partial derivative $\partial_1 \partial_2 ... \partial_{d-1} C(\mathbf{x}, y)$ does not exist.

Studying Archimedean copulas exhibiting this pathological behavior, we exploit the one-to-one correspondence between Archimedean copulas and their associated Williamson measures. Using this correspondence, it becomes apparent that point masses of the Williamson measure $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ lead to a lack of differentiability of the corresponding Archimedean copula $C_\gamma \in \mathcal{C}^d_{ar}$. More precisely, the existence of some $z_0 \in (0, \infty)$ with $\gamma(\{z_0\}) > 0$ implies $C_\gamma \in \mathcal{C}^d_{ar,p}$. We explain this connection in more detail since this observation will be crucial throughout this section and start with the following lemma:

Lemma 5.1. *Let $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ and let $C \in \mathcal{C}^d_{ar}$ be the associated Archimedean copula. If the Williamson measure γ has a point mass, that is, if there exists some $z_0 \in (0, \infty)$ such that $\gamma(\{z_0\}) > 0$, then for every $\mathbf{x} \in [C]_{t_0} \cap (0,1)^{d-1}$ and $\mathbf{x} \notin L_0^{1:d-1}$ the partial derivative $\partial_1 \cdots \partial_{d-1} C(\mathbf{x}, y_{\mathbf{x}})$ does not exist, where $t_0 := \psi(1/z_0)$ and $y_{\mathbf{x}} := f^{t_0}(\mathbf{x})$.*

In other words, whenever γ has a point mass, the copula C_γ fails to be differentiable on the set $\Gamma(f^{t_0}) \cap (0,1)^d$.

The same result still holds for any ordering of the mixed partial derivatives of order $(d-1)$.

Proof. Applying d-monotonicity of ψ yields that the partial derivatives up to order $d-2$ exist and therefore the following identity holds for every $\mathbf{x} \in (0,1)^{d-1}$ and $y \in (0,1)$:

$$\partial_2 ... \partial_{d-1} C(\mathbf{x}_{1:d-1}, y) = \prod_{j=2}^{d-1} \varphi'(x_j) \psi^{(d-2)} \left(\sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y) \right). \tag{22}$$

Assume there exists some $z_0 \in (0, \infty)$ such that $\gamma(\{z_0\}) > 0$, and define $t_0 := \psi(1/z_0)$. Fix $\mathbf{x} \in [C^{1:d-1}]_{t_0} \cap (0,1)^{d-1}$ when $t_0 > 0$, and assume that $\mathbf{x} \notin L_{t_0}^{1:d-1}$ and $M(\mathbf{x}) < 1$ when $t_0 = 0$. We then define

$$y := y_{\mathbf{x}} := f^{t_0}(\mathbf{x}).$$

Substituting this choice of y into equation (22) and invoking the convexity of $(-1)^{d-2}\psi^{(d-2)}$, we obtain

$$\begin{aligned} \partial_1^- \partial_2 \cdots \partial_{d-1} C(\mathbf{x}, y) &= \prod_{i=1}^{d-1} \varphi'(x_i) \, D^- \psi^{(d-2)} \left(\sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y) + \right) \\ &= \prod_{i=1}^{d-1} \varphi'(x_i) \, D^- \psi^{(d-2)} (\varphi(t_0)+), \end{aligned}$$

as well as

$$\partial_1^+\partial_2\cdots\partial_{d-1}C(\mathbf{x},y) = \prod_{i=1}^{d-1}\varphi'(x_i)\,D^-\psi^{(d-2)}(\varphi(t_0))\,.$$

Since z_0 was a point mass of γ, according to Lemma 3.7 the point $\frac{1}{z_0} = \varphi(t_0)$ is a point of discontinuity of $D^-\psi^{(d-2)}$. This yields

$$\partial_1^+\partial_2...\partial_{d-1}C(\mathbf{x},y) \neq \partial_1^-\partial_2...\partial_{d-1}C(\mathbf{x},y).$$

Applying similar arguments as above, the same result holds for any ordering of the mixed partial derivatives of order $(d-1)$. ☐

The next result extends the previous observations. Given the close relationship between the regularity of the Williamson measure $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ and the regularity of Archimedean copulas, this result comes as no surprise. In particular, we will show in the next section that elements of $\mathcal{C}^d_{ar,\mathcal{Q}}$ (and indeed even elements of $\mathcal{C}^d_{ar,p}$) are highly atypical in the sense of Baire categories. Before examining the topological sizes of subsets of $\mathcal{C}^d_{ar}$, we first show that $\mathcal{C}^d_{ar,p}$ consists exclusively of Archimedean copulas whose Williamson measures have a non-degenerate discrete component.

Theorem 5.2. *For $C \in \mathcal{C}^d_{ar}$ the following three assertions are equivalent:*

1. *There exists some $z_0 \in (0,\infty)$ such that $\gamma(\{z_0\}) > 0$.*
2. *$\mu_C^{dis}(\mathbb{I}^d) > 0$, i.e., C has non degenerate discrete component.*
3. *$C \in \mathcal{C}^d_{ar,p}$.*

Proof. We consider the case $d \geq 3$, the case $d = 2$ is technically simpler and can be proved analogously. The equivalence of the first and the second assertion has already been established in Corollary 3.8. The fact that the first assertion implies the third assertion has been proved in Lemma 5.1.

We show that the third assertion implies the second assertion and proceed as follows. If $C \in \mathcal{C}^d_{ar,p}$ then there exists some set $\Lambda \in \mathcal{B}(\mathbb{I}^{d-1})$ with $\mu_{C^{1:d-1}}(\Lambda) > 0$ such that for every point $\mathbf{x} \in \Lambda$ there exists some $y_{\mathbf{x}} \in (0,1)$ with $\partial_1^+\partial_2...\partial_{d-1}C(\mathbf{x},y_{\mathbf{x}}) \neq \partial_1^-\partial_2...\partial_{d-1}C(\mathbf{x},y_{\mathbf{x}})$. Explicitly calculating these derivatives as before it follows that

$$D^-\psi^{(d-2)}\left(\sum_{i=1}^{d-1}\varphi(x_i) + \varphi(y_{\mathbf{x}})\right) \neq D^-\psi^{(d-2)}\left(\sum_{i=1}^{d-1}\varphi(x_i) + \varphi(y_{\mathbf{x}}) +\right),$$

which, using equation (16) yields $K_C(\mathbf{x},\{y_{\mathbf{x}}\}) > 0$. Applying disintegration and equation (3) the property $\mu_C^{dis}(\mathbb{I}^d) > 0$ follows. ☐

We start with the following simple example of an element in $\mathcal{C}^3_{ar,p}$ illustrating the ideas for the general setting.

Example 5.3. Considering the Williamson measure $\gamma = \frac{32}{49}\,\delta_{\frac{1}{8}} + \frac{17}{49}\,\delta_2$ the Williamson 3-transform (7) yields the non-strict generator $\psi\colon [0,\infty) \to \mathbb{I}$, given by

$$\psi(z) = \begin{cases} \frac{137z^2-152z+98}{98}, & \text{if } z \in [0, \frac{1}{2}], \\ \frac{32}{49}(1-\frac{z}{8})^2, & \text{if } z \in (\frac{1}{2}, 8], \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.3 implies that the graphs of the functions f^a with $a = 0 = \psi(8)$ and $a = \psi(2) = \frac{225}{392}$ have mass $\frac{32}{49}$ and $\frac{17}{49}$, respectively. According to Lemma 5.1 we obtain that for ever $\mathbf{x} \in [C^{1:2}]_a \cap (0,1)^2$ with $\mathbf{x} \notin L_0^{1:2}$ there exists a $y_{\mathbf{x}} := f^a(\mathbf{x})$ such that the mixed partial derivative $\partial_1\partial_2 C(\mathbf{x}, y_{\mathbf{x}})$ does not exist. The graphs of the functions f^a with $a \in \{0, \frac{225}{392}\}$ are depicted in Figure 3. These contain the points $(\mathbf{x}, y) \in \mathbb{I}^3$ where the derivative $\partial_1\partial_2 C(\mathbf{x}, y)$ does not exist.

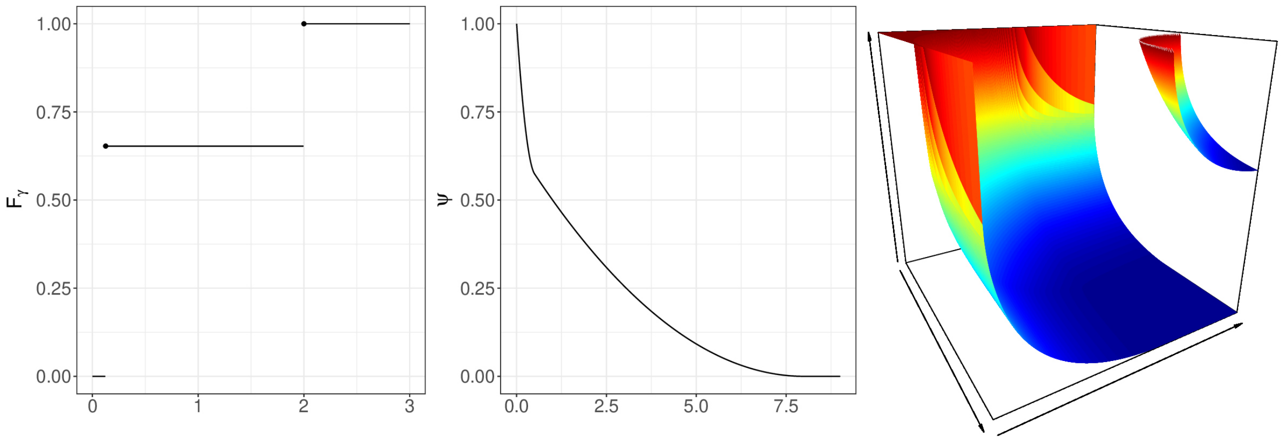

Figure 3: Plots of the distribution function F_γ (left panel) of the Williamson measure γ considered in Example 5.3, the associated generator ψ (middle) and the graphs of the functions f^a with $a \in \{0, \frac{225}{392}\}$ of the corresponding Archimedean copula C (right panel).

We now focus on the class $\mathcal{C}^d_{ar,\mathcal{Q}}$ and show that it is non-empty.

Theorem 5.4 (Non-differentiability on dense subset)**.** *There are copulas $C \in \mathcal{C}^d_{ar}$ with the following property: there is some set $\Lambda \in \mathcal{B}(\mathbb{I}^{d-1})$ with $\lambda_{d-1}(\Lambda) = 1$ such that for every $\mathbf{x} \in \Lambda$ it holds that $\partial_1 \ldots \partial_{d-1} C(\mathbf{x}, y)$ does not exist for a dense set of $y \in \mathbb{I}$.*
The same result still holds for any ordering of the mixed partial derivatives of order $(d-1)$.

Proof. We prove the result for $d \geq 3$, the case $d = 2$ can be proved similarly. Suppose that $Q = \{q_1, q_2, \ldots\} \subseteq (0,\infty)$ is dense in $[0,\infty)$, that $\alpha_1, \alpha_2, \ldots \in (0,1)$ fulfill $\sum_{i=1}^{\infty} \alpha_i = 1$, and that

$$\gamma := \sum_{i\in\mathbb{N}} \alpha_i \delta_{q_i} \in \mathcal{P}_{\mathcal{W}_d}.$$

holds. For ways to construct Williamson measures of this type see, e.g., [15, Lemma Appendix B.2]. Fix $\mathbf{z} \in (0,1)^{d-1}$ with $z_k \in (0,1) \setminus \{\psi(\frac{1}{q_j}) : j \in \mathbb{N}\}$ for every $k \in \{1, ..., d-1\}$ and set $y_i = f^{\psi(\frac{1}{q_i})}(\mathbf{z})$ for all $i \in \mathbb{N}$ with $\frac{1}{q_i} > \sum_{k=1}^{d-1} \varphi(z_k)$. Notice that in this case we have $\mathbf{z} \in [C^{1:d-1}]_{\psi\left(\frac{1}{q_i}\right)}$, so y_i is well-defined. Applying Lemma 5.1 yields that $s \mapsto \partial_2 ... \partial_{d-1} C(s, \mathbf{z}_{2:d-1}, y_i)$ is not differentiable in z_1. Since the set

$$\left\{ f^{\psi(\frac{1}{q_j})}(\mathbf{z}) \colon \frac{1}{q_j} > \sum_{k=1}^{d-1} \varphi(z_k), j \in \mathbb{N} \right\}$$

obviously is dense in $\mathbb{I}$, the main result is proved.
Applying similar arguments and using Lemma 5.1, the same result still holds for any ordering of the mixed partial derivatives of order $(d-1)$. □

The next result shows that elements of $\mathcal{C}^d_{ar,\mathcal{Q}}$ can be found in every open ball of the compact metric space $(\mathcal{C}^d_{ar}, d_\infty)$.

Corollary 5.5. *The set* $\mathcal{C}^d_{ar,\mathcal{Q}}$ *is dense in* $(\mathcal{C}^d_{ar}, d_\infty)$.

Proof. Fix $C \in \mathcal{C}^d_{ar}$ and let $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ be its corresponding Williamson measure. Then according to Lemma Appendix B.1 there exists a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of discrete Williamson measures with full support which converges weakly to γ. Using Theorem 5.4 every γ_n induces a copula $C_n \in \mathcal{C}^d_{ar,\mathcal{Q}}$, so applying [15, Theorem 5.9] or Lemma 2.2 yields the desired result. □

6. Baire category results for multivariate Archimedean copulas

Building upon the previous sections on mass distributions and derivatives, we now study which regularity properties topologically typical Archimedean copulas exhibit. To this end, we establish Baire category results for the family $(\mathcal{P}_{\mathcal{W}_d}, \tau_w)$ of all d-Williamson measures (see Appendix B) and then transfer/translate these results to $\mathcal{C}^d_{ar}$, using the fact that homeomorphisms preserve Baire categories (see, e.g., [3, Lemma A1]). Recall that the space of all Archimedean copulas $(\mathcal{C}^d_{ar}, d_\infty)$ is not complete, so it is a priori unclear whether $\mathcal{C}^d_{ar}$ is not of first Baire category in itself. To facilitate readability, all auxiliary lemmas on Baire categories on $(\mathcal{P}_{\mathcal{W}_d}, \tau_w)$ are gathered and proved in Appendix B. As mentioned in Section 2.2, the family $\mathcal{P}^d_{nor}$ endowed with τ_w is complete and hence closed. Translating results from $\mathcal{P}_{\mathcal{W}_d}$ to $\mathcal{C}^d_{ar}$ then implies that a topologically typical Archimedean copula has full support.

In the following we extend the result from [5], stating that a typical Archimedean copula is strict, to arbitrary dimensions $d \geq 2$. Building upon the tools developed in the previous sections and in [15] the proof provided below is much simpler than the one for the bivariate setting established in [5]. In fact, we are even able to prove a stronger result: A typical multivariate Archimedean copula has full support.

Theorem 6.1. *The set* $\{C \in \mathcal{C}^d_{ar} \colon \mathrm{supp}(\mu_C) = \mathbb{I}^d\}$ *is co-meager in* $(\mathcal{C}^d_{ar}, d_\infty)$.

Proof. Immediate consequence of Lemma 2.2, Theorem 3.6 and Lemma Appendix B.4. □

The subsequent corollary is an immediate consequence of Lemma Appendix B.4:

Corollary 6.2. *The set* $\mathcal{P}^s_{\mathcal{W}_d}$ *is co-meager in* $\mathcal{P}_{\mathcal{W}_d}$ *w.r.t. the weak topology. Moreover,* $\mathcal{C}^d_{ar,s}$ *is co-meager in* $(\mathcal{C}^d_{ar}, d_\infty)$.

Proof. The first assertion follows from the facts that $\mathcal{P}^{fs}_{\mathcal{W}_d} \subseteq \mathcal{P}^s_{\mathcal{W}_d}$ is co-meager in $\mathcal{P}_{\mathcal{W}_d}$ and that supersets of co-meager sets are themselves co-meager. Having this, applying Lemma 2.2, Theorem 3.6 and [15, Lemma 5.5] proves the second assertion. □

As mentioned at the beginning of this section, since the space of d-dimensional Archimedean copulas is not complete, the fact that the families $\mathcal{C}^d_{ar,s}$ and $\{C \in \mathcal{C}^d_{ar} \colon \mathrm{supp}(\mu_C) = \mathbb{I}^d\}$ are of second Baire category is not obvious.
However, using Lemma 2.2 and translating Lemma Appendix B.5 to $(\mathcal{C}^d_{ar}, d_\infty)$, the subsequent result follows:

Corollary 6.3. *The sets* $\mathcal{C}^d_{ar,s}$ *and* $\{C \in \mathcal{C}^d_{ar} \colon \mathrm{supp}(\mu_C) = \mathbb{I}^d\}$ *are of second Baire category in* $(\mathcal{C}^d_{ar}, d_\infty)$.

As next step we show that a typical d-dimensional Archimedean copula is not absolutely continuous.

Theorem 6.4. *The family* $\mathcal{C}^d_{ar,abs}$ *is of first Baire category in* $(\mathcal{C}^d_{ar}, d_\infty)$.

Proof. Immediate consequence of the fact that homeomorphisms map meager sets to meager sets, Lemma 2.2, Corollary 3.13 and Lemma Appendix B.6. □

Using Lemma 2.2, Lemma Appendix B.7 and Theorem 3.12 we have shown that typical Archimedean copulas have degenerate discrete component - the following result holds:

Theorem 6.5. *The set*

$$\{C \in \mathcal{C}^d_{ar} \colon \mu^{dis}_C(\mathbb{I}^d) = 0\}$$

is co-meager in $(\mathcal{C}^d_{ar}, d_\infty)$.

Proceeding analogously to the proof of Corollary 6.3 yields the following:

Corollary 6.6. *The set*

$$\{C \in \mathcal{C}^d_{ar} \colon \mu^{dis}_C(\mathbb{I}^d) = 0\}$$

is of second Baire category in $(\mathcal{C}^d_{ar}, d_\infty)$.

Remark 6.7. Considering Theorem 5.2 and Theorem 6.5 we conclude that topologically typical Archimedean copulas do not exhibit pathological behavior, i.e., a typical d-dimensional Archimedean copula C fulfills $C \notin \mathcal{C}^d_{ar,p}$.

Combining Theorem 6.1, Theorem 6.4 and Theorem 6.5 yields the following surprising main result on topologically typical multivariate Archimedean copulas:

Corollary 6.8. *A topologically typical d-dimensional Archimedean copula C has full support, has degenerate discrete component and is not absolutely continuous.*

We conclude this section with an example of a topologically atypical bivariate Archimedean copula.

Example 6.9. Consider the 2-Williamson measure $\gamma \in \mathcal{P}_{\mathcal{W}_2}$, defined via its distribution function by

$$\gamma([0,z]) := \begin{cases} 0, & \text{if } z \in [0, \frac{1}{4}), \\ \frac{2}{3}, & \text{if } z \in [\frac{1}{4}, 1), \\ \frac{1}{3}(\frac{1}{4}\sqrt{z-1} + 2), & \text{if } z \in [1,2), \\ \frac{1}{8}(\sqrt[3]{z-2} + 7), & \text{if } z \in [2,3), \\ 1, & \text{if } z \geq 3, \end{cases}$$

for every $z \in [0, \infty)$. Obviously (see Figure 4) γ has a non-degenerate discrete as well as a non-degenerate absolutely continuous component. Straightforward calculations show that the corresponding generator ψ is given by

$$\psi(z) = \begin{cases} 1 - \frac{233z}{288}, & \text{if } z \in [0, \frac{1}{3}), \\ \frac{27\sqrt[3]{(1-2z)^4} - 4\sqrt[3]{z}(38z-63)}{288\sqrt[3]{z}}, & \text{if } z \in (\frac{1}{3}, \frac{1}{2}], \\ \frac{-3\sqrt{z^3} + \sqrt{(1-z)^3} + 12\sqrt{z}}{18\sqrt{z}}, & \text{if } z \in (\frac{1}{2}, 1], \\ \frac{2}{3}(1 - \frac{z}{4}), & \text{if } z \in (1, 4], \\ 0, & \text{if } z > 4. \end{cases}$$

The induced Archimedean copula C_ψ has non-degenerate discrete component and is non-strict, so according to Corollary 6.8 the copula C_ψ is atypical. A sample of the copula C_ψ is depicted in Figure 5.

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Declaration

Conflict of interest. The authors have no competing interests to declare that are relevant to the content of this article.

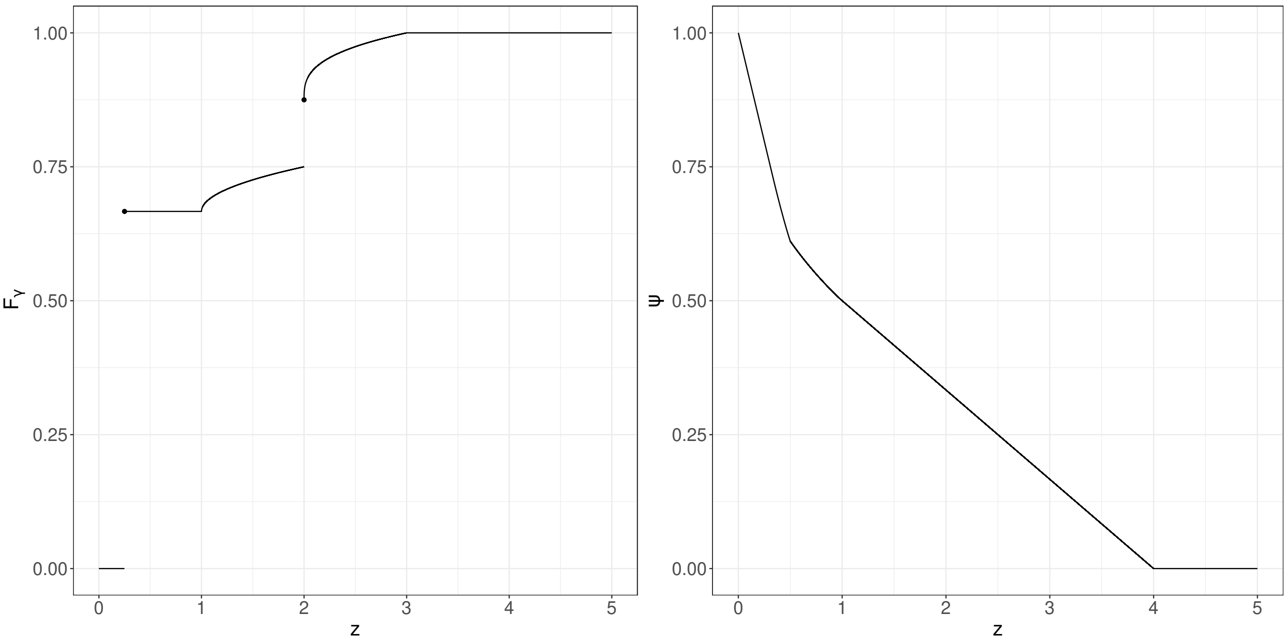

Figure 4: The distribution function F_γ (left panel) and the associated generator ψ (right panel) considered in Example 6.9.

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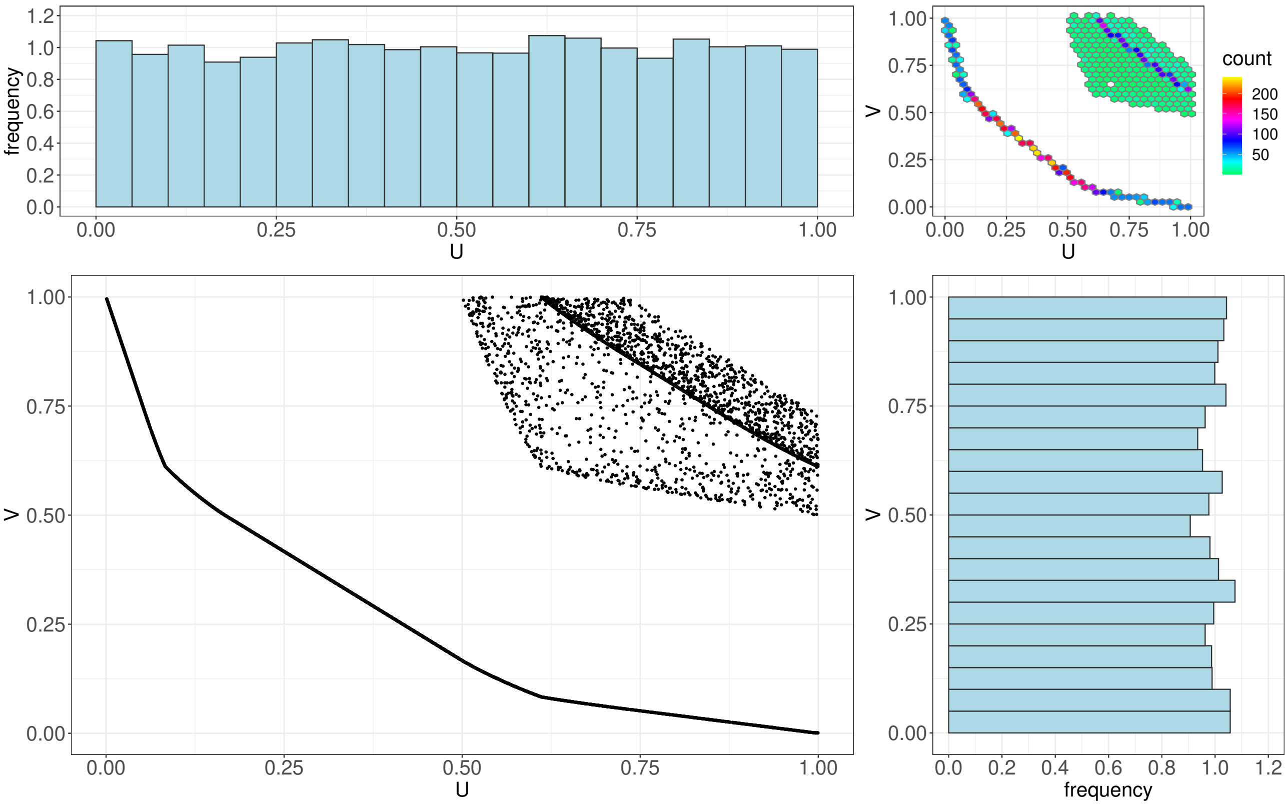

Figure 5: Sample of size 10.000 of the Archimedean copula C_ψ with ψ being the generator from Example 6.9, its histogram and the two marginal histograms; sample generated via conditional inverse sampling.

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Appendix A. Proofs for auxiliary results in Sections 3,4 and 5

Appendix A contains the proofs of several technical lemmas, which have been moved here in order to improve the readability of the paper.

Proof of Lemma 3.7. The equivalence of the first, second and fourth assertion is an immediate consequence of Theorem 2.3 and equations (12) and (13), respectively. The fact that the second assertion implies the third one is an immediate consequence of the form of the Markov kernel in equation (16). Applying disintegration proves that the third assertion implies the fourth one, which completes the proof.
Note that alternatively the equivalence between the first, the second and the fourth assertion also follows from the results in Section 4.1 in [21]. □

Proof of Lemma 3.9. We assume that $\gamma^{abs}([0,\infty)) > 0$ and prove that for all $\mathbf{x}$ in a set of positive $\mu_{C^{1:d-1}}$-measure the Markov kernel $K_C(\mathbf{x},\cdot)$ has non-degenerate absolutely continuous component. Setting $z_0 := \sup\{z \in [0,\infty)\colon \gamma^{abs}([0,z]) = 0\}$ we have $z_0 \in [0,\infty)$. Define the set $\Lambda \subseteq \mathbb{I}^{d-1}$ by

$$\Lambda := \left\{ \mathbf{x} \in [0,1)^{d-1}\colon 0 < \sum_{i=1}^{d-1} \varphi(x_i) < \frac{1}{z_0} \right\}.$$

Then we obviously have $\lambda_{d-1}(\Lambda) > 0$. Using the equivalence of $\gamma([0,r)) = 0$ and $\psi(\frac{1}{r}) = 0$ for $r \in (0,\infty)$, it follows that $\frac{1}{z_0} \leq \varphi(0)$ holds. Hence for $\mathbf{x} \in \Lambda$ the density $c^{1:d-1}$ of $C^{1:d-1}$ fulfills $c^{1:d-1}(\mathbf{x}) > 0$, which altogether implies that $\mu_{C^{1:d-1}}(\Lambda) > 0$. Therefore, using equation (18) it suffices to show that for fixed $\mathbf{x} \in \Lambda$ the function

$$y \mapsto H^{abs}_{\mathbf{x}}(y) = \frac{\int_{I_y} t^{d-1} \mathrm{d}\gamma^{abs}(t)}{\int_{I_1} t^{d-1} \mathrm{d}\gamma(t)} \tag{A.1}$$

is non-degenerate and absolutely continuous on $\mathbb{I}$, which can be done as follows: For $\mathbf{x} \in \Lambda$ the construction of z_0 implies that for $y_0 = f^0(\mathbf{x})$ we have

$$\frac{1}{\sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y_0)} \leq \frac{1}{\varphi(0)} \leq z_0 < \frac{1}{\sum_{i=1}^{d-1} \varphi(x_i)} = \frac{1}{\sum_{i=1}^{d-1} \varphi(x_i) + \varphi(1)},$$

which, using the fact that $t^{d-1} > 0$ for every $t \in (0,\infty)$, yields that $H_{\mathbf{x}}^{abs}$ is non-degenerate; $H_{\mathbf{x}}^{abs}$ is obviously non-negative, non-decreasing, and continuous on $\mathbb{I}$.
Moreover the function $z \mapsto \int_{[0,z]} t^{d-1}\mathrm{d}\gamma^{abs}(t)$ is absolutely continuous and non-decreasing on every compact interval of the form $[0,a]$. Considering that both, in the strict and in the non-strict case, we have that the mapping $y \mapsto \frac{1}{\sum_{i=1}^{d-1}\varphi(x_i)+\varphi(y)}$ is absolutely continuous and non-decreasing on $\mathbb{I}$, according to [24, Proposition 129] the composition $H_{\mathbf{x}}^{abs}$ is absolutely continuous too. Finally, equation (19) implies that $K_C(\mathbf{x},\cdot)$ has a non-degenerate absolutely continuous component, which completes the proof. $\square$

Proof of Lemma 3.10. We assume $\gamma^{sing}([0,\infty)) > 0$ and proceed in the same manner as for the absolutely continuous component. Define the set Λ in the same manner as in the proof of Lemma 3.9 only replacing γ^{abs} by γ^{sing} in the definition of z_0. It suffices to show that for fixed $\mathbf{x} \in \Lambda$ the function

$$y \mapsto H_{\mathbf{x}}^{sing}(y) = \frac{\int_{I_y} t^{d-1}\mathrm{d}\gamma^{sing}(t)}{\int_{I_1} t^{d-1}\mathrm{d}\gamma(t)} \tag{A.2}$$

is non-degenerate and singular on $\mathbb{I}$, which according to equation (19) implies that the Markov kernel $K_C(\mathbf{x},\cdot)$ has a non-degenerate singular component.
The construction of Λ implies that $H_{\mathbf{x}}^{sing}$ is non-degenerate. Moreover $H_{\mathbf{x}}^{sing}$ is continuous since discontinuity points would correspond to point masses of γ^{sing}, of which by assumption there are none.
In order to prove singularity of $H_{\mathbf{x}}^{sing}$ let $\mathbf{x} \in \Lambda$ be arbitrary but fixed and proceed as follows: Define the σ-finite measure m on $\mathcal{B}([0,\infty))$ by

$$m(E) := \int_E t^{d-1}\mathrm{d}\gamma^{sing}(t).$$

Then there exists some set $O \in \mathcal{B}([0,\infty))$ with $m([0,\infty)\setminus O) = 0$ and $\lambda(O) = 0$. Letting $G_m\colon I_1 \to [0,\infty)$ denote the induced measure-generating function (restricted to the set I_1), defined by $G_m(z) := m([0,z])$, singularity of m implies that $G_m'(z) = 0$ for λ-almost every $z \in I_1$, i.e., setting

$$F := \{z \in I_1\colon G_m' \text{ exists and } G_m'(z) = 0\}$$

we have $\lambda(F) = \lambda(I_1)$. The function $g : \mathbb{I} \to \left[\frac{1}{\sum_{k=1}^{d-1}\varphi(x_k)+\varphi(0)}, \frac{1}{\sum_{k=1}^{d-1}\varphi(x_k)}\right] =: I_{0,1}$, defined by $g(y) := \frac{1}{\sum_{i=1}^{d-1}\varphi(x_i)+\varphi(y)}$ obviously is strictly increasing, continuous and bijective. Let h denote its inverse and set

$$\Upsilon := \{y \in \mathbb{I}\colon g(y) \in F\} = g^{-1}(F) \in \mathcal{B}(\mathbb{I}).$$

Then for every $z \in I_{0,1}$ we have $h(z) = \psi(\frac{1}{z} - \sum_{i=1}^{d-1}\varphi(x_i))$ as well as $h(F) = \Upsilon$. Using convexity of ψ yields that h is locally Lipschitz continuous and therefore locally absolutely continuous. Applying that locally absolutely continuous functions map sets of λ-measure 0 to sets of λ-measure 0 [20, Theorem

3.41] implies that $\lambda(\mathbb{I} \setminus \Upsilon) = 0$ holds. For fixed $y \in \Upsilon \cap (0,1)$, using equation (A.1) and applying the chain rule yields

$$\begin{aligned}(H^{sing})'(y) &= \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}y}G_m\left(\frac{1}{\sum_{i=1}^{d-1}\varphi(x_i)+\varphi(y)}\right)\\ &= \frac{1}{c}G_m'\left(\frac{1}{\sum_{i=1}^{d-1}\varphi(x_i)+\varphi(y)}\right)\frac{-D^+\varphi(y)}{(\sum_{i=1}^{d-1}\varphi(x_i)+\varphi(y))^2} = 0,\end{aligned}$$

with $c = \int_{I_1} t^{d-1}\mathrm{d}\gamma(t)$. Since $\lambda(\Upsilon \cap (0,1)) = 1$ we have shown that $H_{\mathbf{x}}^{sing}$ has derivative zero λ-almost everywhere in $\mathbb{I}$, which completes the proof. $\square$

Proof of Lemma 4.1. As already mentioned in Section 2 for every $r \in (0,\infty)$ we have $\gamma([0,r)) = 0$ if, and only if $\psi(\frac{1}{r}) = 0$. This directly yields $\gamma([0,z]) = 0$ for every $z \in [0, \frac{1}{\varphi(0)})$.
On the other hand, for $z \geq \frac{1}{\varphi(0)}$ there exists a unique $t \in \mathbb{I}$ with $\varphi(t) = \frac{1}{z}$, and this t coincides with $\psi(\frac{1}{z})$. Applying equation (15) the desired identity $\gamma([0,z]) = F_K^d(\psi(\frac{1}{z}))$ follows.
The second assertion is a direct consequence of the facts that (i) ψ is a generator and (ii) the mapping $z \mapsto \frac{1}{z}$ is Lipschitz continuous on every interval of the form $[a,1]$ with $a > 0$. $\square$

Appendix B. Denseness and Baire category results for probability measures on $[0,\infty)$

In this appendix, we establish denseness and Baire category properties for probability measures on the interval $[0,\infty)$, with a particular focus on Williamson measures. These findings are essential for the Baire category results regarding Archimedean copulas in Section 6.
In order to show that fully supported absolutely continuous, discrete and singular Williamson measures are dense in $(\mathcal{P}_{\mathcal{W}_d}, \tau_w)$ we make use of the subsequent lemma.

Lemma Appendix B.1. *The following assertions hold:*

(i) The family of discrete d-Williamson measures with full support is dense in $(\mathcal{P}_{\mathcal{W}_d}, \tau_w)$.

(ii) The family of singular d-Williamson measures with full support is dense in $(\mathcal{P}_{\mathcal{W}_d}, \tau_w)$.

(iii) The family of absolutely continuous d-Williamson measures with full support is dense in $(\mathcal{P}_{\mathcal{W}_d}, \tau_w)$.

Proof. Fix $\gamma \in \mathcal{P}_{\mathcal{W}_d}$. Then according to [15, Theorem Appendix B.2] there exists a sequence $(\gamma_n)_{n\in\mathbb{N}}$ of absolutely continuous/discrete/singular Williamson measures converging weakly to γ. Considering an absolutely continuous/discrete/singular Williamson measure β with full support (see, e.g., [15, Theorem 6.1] and [15, Theorem 6.2] for examples of such measures) and setting $\beta_n := (1-\frac{1}{n})\gamma_n + \frac{1}{n}\beta$ for every $n \in \mathbb{N}$ yields a sequence $(\beta_n)_{n\in\mathbb{N}}$ of absolutely continuous/discrete/singular Williamson measures with full support which converges weakly to γ. $\square$

Next, we show that the family of all d-Williamson measures $\mathcal{P}_{\mathcal{W}_d}$ is dense in $(\mathcal{P}^d_{nor}, \tau_w)$.

Lemma Appendix B.2. *The family of all d-Williamson-measures $\mathcal{P}_{\mathcal{W}_d}$ is dense in $(\mathcal{P}^d_{nor}, \tau_w)$.*

Proof. First of all notice that every $\gamma \in \mathcal{P}^d_{nor}$ fulfills $\gamma(\{0\}) \leq \frac{1}{2}$. Since for $\gamma(\{0\}) = 0$ we have $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ it suffices to consider the case $\gamma(\{0\}) \in (0, \frac{1}{2}]$ and to show that there exists some sequence in $\mathcal{P}_{\mathcal{W}_d}$ converging weakly to γ.
(i) Suppose that $0 < \gamma(\{0\}) < \frac{1}{2}$ holds. Then we have $\gamma(\mathbb{I}) > \frac{1}{2}$ and may proceed as follows: For $a \in [0, \frac{1}{4}]$ and $b \in [a, 1]$ define the transformation $T_{a,b} \colon \mathbb{I} \to [a, b]$ by $T_{a,b}(t) := a + (b - a)t$. Choose $n_0 \in \mathbb{N}$ sufficiently large so that $\gamma(\mathbb{I})(1 - \frac{1}{n_0})^{d-1} > \frac{1}{2}$ holds and set $a_n := \frac{1}{n}$ for every $n \geq n_0$. Defining the function $\Phi_n : [a_n, 1] \to [0, \infty)$ by

$$\Phi_n(s) := \int_{\mathbb{I}} (1 - a_n - (s - a_n)t)^{d-1} \mathrm{d}\gamma(t) = \int_{[a_n,s]} (1 - t)^{d-1} \mathrm{d}\gamma^{T_{a_n,s}}(t)$$

it follows that Φ_n is strictly decreasing and continuous on $[a_n, 1]$ and that $\Phi_n(a_n) \geq \frac{1}{2}$ as well as $\Phi_n(1) \leq \frac{1}{2}$ hold. Hence there exists a unique $b_n \in [a_n, 1]$ fulfilling $\Phi(b_n) = \frac{1}{2}$. Letting ξ_n denote the unique probability measure on $\mathcal{B}([0, \infty))$ fulfilling $\xi_n([0, a_n)) = 0 = \xi_n((b_n, 1])$, coinciding with $\gamma^{T_{a_n,b_n}}$ on $\mathcal{B}([a_n, b_n])$ and with γ on $\mathcal{B}((1, \infty))$ we obviously have $\xi_n \in \mathcal{P}_{\mathcal{W}_d}$ for every $n \geq n_0$. Considering $a_n \overset{n\to\infty}{\longrightarrow} 0$ it is straightforward to verify that $b_n \overset{n\to\infty}{\longrightarrow} 1$. Hence, for every bounded continuous function $f \colon [0, \infty) \to \mathbb{R}$ using change of coordinates it follows that

$$\begin{aligned}
\int_{[0,\infty)} f(t)\mathrm{d}\xi_n(t) &= \int_{[0,a_n)\cup(b_n,1]} f(t)\mathrm{d}\xi_n(t) \\
&\quad + \int_{[a_n,b_n]} f(t)\mathrm{d}\xi_n(t) + \int_{(1,\infty)} f(t)\mathrm{d}\xi_n(t) \\
&= 0 + \int_{[a_n,b_n]} f(t)\mathrm{d}\gamma^{T_{a_n,b_n}}(t) + \int_{(1,\infty)} f(t)\mathrm{d}\xi_n(t) \\
&= \int_{\mathbb{I}} f \circ T_{a_n,b_n}(t)\mathrm{d}\gamma(t) + \int_{(1,\infty)} f(t)\mathrm{d}\gamma(t)
\end{aligned}$$

and the latter sum converges to $\int_{[0,\infty)} f(t)\mathrm{d}\gamma(t)$. In other words: the sequence $(\xi_n)_{n\in\mathbb{N}}$ converges weakly to γ.
(ii) Suppose that $\gamma(\{0\}) = \frac{1}{2}$ holds, in which case we have $\gamma(\mathbb{I}) = \frac{1}{2}$. Letting F_γ denote the distribution function of γ and F^-_γ its quasi-inverse then we obviously have $F^-_\gamma(\frac{1}{2} + \frac{1}{n}) > 1$. For every $n \geq 3$ define another distribution function F_{ζ_n} by

$$F_{\zeta_n}(z) := \left(\frac{1}{2} + \frac{1}{n}\right) \mathbf{1}_{[1-\sqrt[d-1]{\frac{n}{n+2}}, F^-_\gamma(\frac{1}{2}+\frac{1}{n}))}(z) + F_\gamma(z)\mathbf{1}_{[F^-_\gamma(\frac{1}{2}+\frac{1}{n}),\infty)}(z)$$

for every $z \in [0, \infty)$. Then obviously the probability measures ζ_n induced by F_{ζ_n} is a d-Williamson measure for every $n \geq 3$. If $z \in (0, \infty)$ fulfills $F_\gamma(z) = \frac{1}{2}$, then $z < F^-_\gamma(\frac{1}{2} + \frac{1}{n})$, hence, using the facts that $\sqrt[d-1]{\frac{n}{n+2}} \overset{n\to\infty}{\longrightarrow} 1$ and $\frac{1}{2}+\frac{1}{n} \overset{n\to\infty}{\longrightarrow} \frac{1}{2}$ shows convergence of $(F_{\zeta_n}(z))_{n\in\mathbb{N}}$ to $F_\gamma(z)$. If $z \in (0,\infty)$ fulfills $F_\gamma(z) > \frac{1}{2}$, then we may choose $K \in \mathbb{N}$ sufficiently large so that $\frac{1}{2}+\frac{1}{K} \leq F_\gamma(z)$ and $F_\gamma^-(\frac{1}{2}+\frac{1}{K}) \leq z$ hold. Then $F_{\zeta_n}(z) = F_\gamma(z)$ for all $n \geq K$, which completes the proof. ☐

Building upon the previous lemma we prove that a typical element of $\mathcal{P}^d_{nor}$ is a Williamson measure.

Lemma Appendix B.3. *The family of d-Williamson-measures* $\mathcal{P}_{\mathcal{W}_d}$ *is co-meager in* $(\mathcal{P}^d_{nor}, \tau_w)$.

Proof. It suffices to show that the set

$$N := \{\gamma \in \mathcal{P}^d_{nor} \colon \gamma(\{0\}) > 0\} \tag{B.1}$$

is of first Baire category in $\mathcal{P}^d_{nor}$, which can be done as follows. For every $k \geq 2$ defining the set $\mathcal{P}_k$ by

$$\mathcal{P}_k := \left\{\gamma \in \mathcal{P}^d_{nor} \colon \gamma(\{0\}) \geq \frac{1}{k}\right\}$$

we obviously have $N \subseteq \bigcup_{k=2}^\infty \mathcal{P}_k$. Portmanteau's theorem implies that for every sequence $\gamma_1, \gamma_2, ... \in \mathcal{P}_k$ converging weakly to some $\gamma \in \mathcal{P}^d_{nor}$ we have that $\gamma(\{0\}) \geq \limsup_{n\to\infty} \gamma_n(\{0\}) \geq \frac{1}{k}$ holds, so $\mathcal{P}_k$ is weakly closed in $\mathcal{P}^d_{nor}$. Letting $\mathcal{O} \subseteq \mathcal{P}^d_{nor}$ denote a non-empty open set then denseness of $\mathcal{P}_{\mathcal{W}_d}$ in $\mathcal{P}^d_{nor}$ yields the existence of a measure $\tilde{\gamma} \in \mathcal{O} \cap \mathcal{P}_{\mathcal{W}_d}$ with $\tilde{\gamma} \notin \mathcal{P}_k$. This shows that $\mathcal{P}_k$ is nowhere dense in $\mathcal{P}^d_{nor}$, so the set N is of first Baire category, implying that $\mathcal{P}_{\mathcal{W}_d}$ is co-meager in $\mathcal{P}^d_{nor}$. ☐

The next lemma shows that a typical Williamson measure has full support.

Lemma Appendix B.4. *The set* $\mathcal{P}^{fs}_{\mathcal{W}_d}$ *is co-meager in* $(\mathcal{P}_{\mathcal{W}_d}, \tau_w)$.

Proof. It suffices to show that

$$\mathcal{W} := \{\gamma \in \mathcal{P}_{\mathcal{W}_d} \colon \text{there exists an interval } (a,b) \subseteq [0,\infty) \text{ with } \gamma((a,b)) = 0\}$$

is of first Baire category in $(\mathcal{P}_{\mathcal{W}_d}, \tau_w)$. Let $q_1, q_2, \ldots$ be an enumeration of $(0,\infty) \cap \mathbb{Q}$, set $q_0 := 0$, and, for $(i,k) \in \mathbb{N}_0 \times \mathbb{N}$ define the sets $\mathcal{W}_{i,k}$ by

$$\mathcal{W}_{i,k} := \begin{cases} \{\gamma \in \mathcal{P}_{\mathcal{W}_d} \colon \gamma((q_i - \frac{1}{k}, q_i + \frac{1}{k})) = 0\} \text{ if } i \geq 1, \\ \{\gamma \in \mathcal{P}_{\mathcal{W}_d} \colon \gamma((0, \frac{1}{k})) = 0\} \text{ if } i = 0. \end{cases}$$

Then we obviously have $\mathcal{W} \subseteq \bigcup_{i\in\mathbb{N}_0} \bigcup_{k\in\mathbb{N}} \mathcal{W}_{i,k}$. We show that for $(i,k) \in \mathbb{N}^2$ the set $\mathcal{W}_{i,k}$ is weakly closed. If $(\gamma_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{W}_{i,k}$ converging weakly to some $\gamma \in \mathcal{P}_{\mathcal{W}_d}$, then Portmanteau's theorem shows that

$$0 = \liminf_{\ell\to\infty} \gamma_\ell((q_i - \tfrac{1}{k}, q_i + \tfrac{1}{k})) \geq \gamma((q_i - \tfrac{1}{k}, q_i + \tfrac{1}{k})) \geq 0,$$

so $\gamma \in \mathcal{W}_{i,k}$. Proceeding analogously yields that the sets $\mathcal{W}_{0,k}$ are weakly closed too. Considering that, according to Lemma Appendix B.1, the set of Williamson measures with full support is dense in $\mathcal{P}_{\mathcal{W}_d}$ it follows that each set $\mathcal{W}_{i,k}$ is nowhere dense in $\mathcal{P}_{\mathcal{W}_d}$, which implies that $\mathcal{W}$ is of first Baire category. Hence, by definition $\mathcal{P}^{fs}_{\mathcal{W}_d}$ is co-meager in $(\mathcal{P}_{\mathcal{W}_d}, \tau_w)$. ☐

Next, we prove that the set of all Williamson measures with full support (and therefore also the set of all strict Williamson measures) is of second Baire category in $\mathcal{P}_{\mathcal{W}_d}$.

Lemma Appendix B.5. *The sets* $\mathcal{P}_{\mathcal{W}_d}^{fs}$ *and* $\mathcal{P}_{\mathcal{W}_d}^{s}$ *are of second Baire category in* $(\mathcal{P}_{\mathcal{W}_d}, \tau_w)$.

Proof. Suppose that $\mathcal{P}_{\mathcal{W}_d}^{fs}$ were of first Baire category in $\mathcal{P}_{\mathcal{W}_d}$. Then, considering that $\mathcal{P}_{nor}^{d}$ contains $\mathcal{P}_{\mathcal{W}_d}$, it would be of first Baire category in $\mathcal{P}_{nor}^{d}$ as well. Applying Lemma Appendix B.3 and Theorem Appendix B.4 yields that $\mathcal{P}_{nor}^{d} \setminus \mathcal{P}_{\mathcal{W}_d}$ and $\mathcal{P}_{\mathcal{W}_d} \setminus \mathcal{P}_{\mathcal{W}_d}^{fs}$ are both of first Baire category in $\mathcal{P}_{nor}^{d}$. Therefore, writing $\mathcal{P}_{nor}^{d} = (\mathcal{P}_{nor}^{d} \setminus \mathcal{P}_{\mathcal{W}_d}) \cup (\mathcal{P}_{\mathcal{W}_d}^{fs} \cup (\mathcal{P}_{\mathcal{W}_d} \setminus \mathcal{P}_{\mathcal{W}_d}^{fs}))$ and using the fact that finite unions of sets of first Baire category are themselves of first Baire category would yield that $\mathcal{P}_{nor}^{d}$ is of first Baire category in itself. A contradiction. Therefore $\mathcal{P}_{\mathcal{W}_d}^{fs}$ is of second Baire category in $\mathcal{P}_{\mathcal{W}_d}$. Considering $\mathcal{P}_{\mathcal{W}_d}^{fs} \subseteq \mathcal{P}_{\mathcal{W}_d}^{s}$ it follows that $\mathcal{P}_{\mathcal{W}_d}^{s}$ is of second Baire category in $\mathcal{P}_{\mathcal{W}_d}$, which completes the proof. $\square$

The subsequent Lemma shows that typical Williamson measures are not absolutely continuous.

Lemma Appendix B.6. *The set* $\mathcal{P}_{\mathcal{W}_d}^{abs}$ *is of first Baire category in* $(\mathcal{P}_{\mathcal{W}_d}, \tau_w)$.

Proof. Suppose that $\gamma \in \mathcal{P}_{\mathcal{W}_d}^{abs}$ has density k_γ; for $n \in \mathbb{N}$ define the sets $M_n^\gamma \in \mathcal{B}([0,\infty))$ by

$$M_n^\gamma := \{z \in [0,\infty) \colon k_\gamma(z) > n\}$$

and set

$$\mathcal{M}_n := \{\gamma \in \mathcal{P}_{\mathcal{W}_d}^{abs} \colon \gamma(M_\gamma^n) \leq \tfrac{1}{4}\}.$$

Considering $\lambda(\bigcap_{n=1}^\infty M_n^\gamma) = 0$, absolute continuity of γ yields $\gamma(\bigcap_{n=1}^\infty M_n^\gamma) = 0$, so for sufficiently large n we have $\gamma(M_n^\gamma) < \frac{1}{4}$. This shows that

$$\mathcal{P}_{\mathcal{W}_d}^{abs} \subseteq \bigcup_{n \in \mathbb{N}} \mathcal{M}_n,$$

it hence suffices to show that $\mathcal{M}_n$ is nowhere dense in $(\mathcal{P}_{\mathcal{W}_d}, \tau_w)$, which can be done as follows: Let $\beta \in \mathcal{P}_{\mathcal{W}_d}^{dis}$ be an arbitrary discrete d-Williamson measure with only finitely many point masses, i.e., $\beta := \sum_{i=1}^N \alpha_i \delta_{x_i}$, whereby $0 < x_1 < x_2 < \cdots < x_N < \infty$ and $\alpha_1, ..., \alpha_N \in (0,1]$ fulfill $\sum_{i=1}^N \alpha_i = 1$. We will show that it is not possible to construct a sequence in $\mathcal{M}_n$ converging weakly to β. Set $x_0 = 0$ and define ρ by

$$\rho := \tfrac{1}{8nN} \min\big\{ \min\{|x_i - x_j| \colon i,j \in \{0,...,N\}\}, 1\big\} > 0.$$

Then obviously we have $\lambda(\bigcup_{i=1}^N (x_i - \rho, x_i + \rho)) \leq \frac{1}{4n}$. Letting $f \colon [0,\infty) \to \mathbb{R}$ denote a continuous function fulfilling $f(x_i) = 1$ and $f\,|_{(x_i-\rho, x_i+\rho)} \in (0,1]$ for every $i \in \{1,\dots,N\}$, and being identical to 0 on $[0,\infty) \setminus \bigcup_{i=1}^N (x_i - \rho, x_i + \rho)$. By construction, on the one hand we have $\int_{[0,\infty)} f \mathrm{d}\beta = 1$. On the other hand, for arbitrary but fixed $\gamma \in \mathcal{M}_n$ it follows that

$$\begin{aligned}\int_{[0,\infty)} f \mathrm{d}\gamma &= \int_{M_n^\gamma} f \mathrm{d}\gamma + \int_{[0,\infty)\setminus M_n^\gamma} f \mathrm{d}\gamma \\ &\leq \frac{1}{4} + \int_{[0,\infty)\setminus M_n^\gamma} f \mathrm{d}\gamma \\ &\leq \frac{1}{2} < 1 = \int_{\mathbb{I}} f \mathrm{d}\beta.\end{aligned}$$

Finally, using the fact that discrete Williamson-measures with finitely many point masses are dense in $(\mathcal{P}_{\mathcal{W}_d}, \tau_w)$ (see [15]) it follows that $\mathcal{M}_n$ is nowhere dense in $\mathcal{P}_{\mathcal{W}_d}$, which proves the desired result. □

We now turn towards Williamson measures having non-degenerate discrete component and show that these elements are topologically atypical.

Lemma Appendix B.7. *The set*

$$\{\gamma \in \mathcal{P}_{\mathcal{W}_d} \colon \gamma \text{ has no point masses}\}$$

is co-meager in $(\mathcal{P}_{\mathcal{W}_d}, \tau_w)$*.*

Proof. It suffices to show that the set

$$\mathcal{P}^p_{\mathcal{W}_d} := \{\gamma \in \mathcal{P}_{\mathcal{W}_d} \colon \exists z \in (0,\infty) \text{ with } \gamma(\{z\}) > 0\}$$

is of first Baire category in $\mathcal{P}_{\mathcal{W}_d}$. For arbitrary $k \in \mathbb{N}$ and $2 \leq m \in \mathbb{N}$ defining the set $\mathcal{W}_{k,m}$ by

$$\mathcal{W}_{k,m} := \left\{\gamma \in \mathcal{P}_{\mathcal{W}_d} \colon \exists z \in \left[\frac{1}{m}, m\right] \text{ such that } \gamma(\{z\}) \geq \frac{1}{k}\right\}$$

we obviously have $\mathcal{P}^p_{\mathcal{W}_d} \subseteq \bigcup_{k=1}^\infty \bigcup_{m=2}^\infty \mathcal{W}_{k,m}$. We will show that $\mathcal{W}_{k,m}$ is closed in $(\mathcal{P}_{\mathcal{W}_d}, \tau_w)$ and proceed as follows: Suppose that $(\gamma_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{W}_{k,m}$ converging weakly to some $\gamma \in \mathcal{P}_{\mathcal{W}_d}$. Then there exists some sequence $(z_n)_{n\in\mathbb{N}}$ in $\left[\frac{1}{m}, m\right]$ fulfilling $\gamma_n(\{z_n\}) \geq \frac{1}{k}$ for every $n \in \mathbb{N}$. Using compactness of $\left[\frac{1}{m}, m\right]$ there exists some subsequence $(z_{n_j})_{j\in\mathbb{N}}$ with limit $z^* \in \left[\frac{1}{m}, m\right]$. We want to prove that $\gamma(\{z^*\}) \geq \frac{1}{k}$ holds. To simplify notation let F_{n_j} and F_γ denote the distribution functions of γ_{n_j} and γ, respectively. Choose an arbitrarily small $h > 0$ such that $z^* + h$ and $z^* - h$ are continuity points of F_γ. Then using weak convergence of $(\gamma_n)_{n\in\mathbb{N}}$ to γ yields that

$$\lim_{j\to\infty} F_{n_j}(z^* + h) = F_\gamma(z^* + h) \text{ and } \lim_{j\to\infty} F_{n_j}(z^* - h) = F_\gamma(z^* - h).$$

Furthermore, convergence of (z_{n_j}) to z^* implies the existence of some $j_0 \in \mathbb{N}$ fulfilling that $|z_{n_j} - z^*| < h$ holds for all $j \geq j_0$. Finally, using monotonicity of distribution functions it follows that

$$\frac{1}{k} \leq \gamma_{n_j}(\{z_{n_j}\}) = F_{n_j}(z_{n_j}) - F_{n_j}(z_{n_j}-) \leq F_{n_j}(z^* + h) - F_{n_j}(z^* - h)$$

holds for sufficiently large j, which implies

$$\frac{1}{k} \leq F_\gamma(z^* + h) - F_\gamma(z^* - h).$$

Since $h > 0$ can be chosen arbitrarily small we finally obtain that $\gamma(\{z^*\}) \geq \frac{1}{k}$, i.e., $\mathcal{W}_{k,m}$ is closed in $(\mathcal{P}_{\mathcal{W}_d}, \tau_w)$. According to Lemma Appendix B.1 the family of absolutely continuous Williamson measures is dense in $(\mathcal{P}_{\mathcal{W}_d}, \tau_w)$, hence $\mathcal{W}_{k,m}$ is nowhere dense in $(\mathcal{P}_{\mathcal{W}_d}, \tau_w)$. This shows that $\mathcal{P}^p_{\mathcal{W}_d}$ is of first Baire category in $\mathcal{P}_{\mathcal{W}_d}$ and the proof is complete. $\square$