Multivariate Archimedean copulas with fractal support

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Abstract

Contributing to the recent literature underlining the fact, that even handy and supposedly smooth families of multivariate copulas may contain elements distributing mass in an unexpected, fairly pathological way, we prove the following result: for every $d \geq 3$ and every $s \in (d-1, d)$ there exists some d-dimensional Archimedean copula A_ψ whose support has Hausdorff dimension s.

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1. Introduction

Being the link between multivariate distribution functions and their univarite marginals, copulas capture all scale-invariant dependence of (continuous) random vectors and have therefore been studied extensively over the past decades (see, e.g., Durante and Sempi (2015)). As Lipschitz continuous distribution functions on $[0,1]^d$, whose univariate marginals correspond to the uniform distribution on $[0,1]$, it might seem natural to conjecture that copulas or, more precisely, their corresponding d-stochastic measures, distribute their mass in fairly regular way to $[0,1]^d$. Fredricks et al. (2005) falsified this interpretation by constructing a family $(A_r)_{r\in(0,\frac{1}{2})}$ of bivariate copulas exhibiting the following property: for every $s \in (1,2)$ there exists some $r_s \in (0, \frac{1}{2})$ such that the Hausdorff dimension of the support of A_{r_s} is s. Since then, various papers on copulas with pathological properties have followed, a natural extension/generalization of the construction by Fredricks et al. (2005) via so-called transformation matrices, e.g., was studied in Trutschnig and Fernández Sánchez (2012).

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Going one step further and considering the family of all bivariate Archimedean copulas (which, being induced by univariate generators, intuitively should exhibit even more regularity), Trutschnig and Fernández Sánchez (2025) showed that the afore-mentioned result by Fredricks et al. also holds in the bivariate Archimedean class. In the current note we tackle the generalization of this very result and establish the following main theorem: For every $d \geq 3$ and arbitrary $s \in (d-1, d)$ there exists some Archimedean copula A_ψ whose support has Hausdorff dimension s.

The remainder of this note is organized as follows: Section 2 contains notation and preliminaries. Section 3 first derives several auxiliary results which then allow to prove our main result for dimension $d = 3$, and sketches the extension to arbitrary dimensions.

2. Notation and preliminaries

For every metric space (S, ρ) the Borel σ-field on S will be denoted by $\mathcal{B}(S)$. We will write λ_d for the d-dimensional Lebesgue measure on $\mathcal{B}(\mathbb{I}^d)$ with $\mathbb{I} = [0,1]$, for $d = 1$ we drop the index and simply write λ instead of λ_1. The topological interior of a set $E \subseteq S$ will be denoted by $int(E)$, the convex hull of $E \subseteq \mathbb{R}^d$ by $\text{convh}(E)$. Furthermore $\mathcal{K}(S)$ will denote the family of all non-empty compact subsets of S, $\mathcal{P}(S)$ the family of all probability measures on $\mathcal{B}(S)$, and $supp(\mu)$ the support of the measure μ (i.e. the the union of all open sets $U \subseteq S$ with $\mu(U) = 0$). It is well-known that $supp(\mu)$ is closed in (S, d).

For every $d \geq 3$ we will let $\mathcal{C}^d$ denote the family of all d-dimensional copulas, i.e., the family of all distribution functions (restricted to $\mathbb{I}^d$) of random vectors $(X_1, \ldots, X_d)$, fulfilling that each X_i is uniformly distributed on $\mathbb{I}$. We use bold symbols for vectors and write $[\mathbf{0}, \mathbf{x}] = [0, x_1] \times [0, x_2] \times \ldots \times [0, x_d]$ for all $(x_1, \ldots, x_d) = \mathbf{x} \in \mathbb{I}^d$. M denotes the minimum copula, i.e. $M(\mathbf{x}) = \min\{x_1, \ldots, x_d\}$. For $C \in \mathcal{C}^d$, the corresponding d-stochastic measure will be denoted by μ_C, i.e. $\mu_C([\mathbf{0}, \mathbf{x}]) = C(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{I}^d$. By definition, the support of a copula C is the support of its d-stochastic measure μ_C. Given $C \in \mathcal{C}^d$ with $d \geq 2$ the marginal copula of the first m coordinates will be denoted by $C^{1:m}$, i.e., for all $(x_1, \ldots, x_m) \in \mathbb{I}^m$ we have $C^{1:m}(x_1, \ldots, x_m) = C(x_1, \ldots, x_m, 1, \ldots, 1)$; for $m = 1$ obviously $C^{1:m}$ is not a copula but coincides with the identity on $\mathbb{I}$. For more background on copulas we refer to Durante and Sempi (2015).

A $(d-1)$-Markov kernel from $\mathbb{R}^{d-1}$ to $\mathbb{R}$ is a mapping $K : \mathbb{R}^{d-1} \times \mathcal{B}(\mathbb{R}) \to \mathbb{R}$ fulfilling that for every fixed $E \in \mathcal{B}(\mathbb{R})$ the transformation $\mathbf{x} \mapsto K(\mathbf{x}, E)$ is $\mathcal{B}(\mathbb{R}^{d-1})-\mathcal{B}(\mathbb{R})$-measurable and for every fixed $\mathbf{x} \in \mathbb{R}^{d-1}$ the mapping $E \mapsto K(\mathbf{x}, E)$ is a probability measure on $\mathcal{B}(\mathbb{R})$. Given a random variable Y and a $(d-1)$-dimensional vector $\mathbf{X}$ on a joint probability space $(\Omega, \mathcal{A}, \mathbb{P})$ we call a Markov kernel $K(\cdot, \cdot)$ a regular conditional distribution of Y given $\mathbf{X}$ if for every fixed $E \in \mathcal{B}(\mathbb{R})$ the identity

$$K(\mathbf{x}(\omega), E) = \mathbb{E}\left(\mathbf{1}_E \circ Y|\mathbf{X}\right)(\omega)$$

holds for $\mathbb{P}$-almost every $\omega \in \Omega$. It is well known that for each pair $(\mathbf{X}, Y)$ as above, a regular conditional distribution of Y given $\mathbf{X}$ exists and is unique for $\mathbb{P}^{\mathbf{X}}$-almost all $\mathbf{x} \in \mathbb{R}^{d-1}$. In the case $(\mathbf{X}, Y)$ has $C \in \mathcal{C}^d$ as distribution function (restricted to $\mathbb{I}^d$) we write $(\mathbf{X}, Y) \sim C$, let $K_C : \mathbb{I}^{d-1} \times \mathcal{B}(\mathbb{I}) \to \mathbb{I}$ denote (a version of) the regular conditional distribution of Y given $\mathbf{X}$, refer to it as $d-1$-Markov kernel of C; notice that we directly consider the kernel K_C as a function from $\mathbb{I}^{d-1} \times \mathcal{B}(\mathbb{I})$ to $\mathbb{I}$. For more background on Markov kernels we refer to Kallenberg (1997), for more results on the interplay between Markov kernels and copulas, e.g., to Kasper et al. (2024); Trutschnig (2011).

An Archimedean generator $\psi : [0, \infty) \to [0, 1]$ is a non-increasing and continuous function which fulfills $\psi(0) = 1$, $\lim_{z\to\infty} \psi(z) = 0$ and which is strictly decreasing on the interval $[0, \inf\{x \in [0, \infty) : \psi(x) = 0\}]$ with the convention $\inf \varnothing = \infty$. The pseudo-inverse φ of an Archimedean generator ψ is the mapping $\varphi : [0, 1] \to [0, \infty]$, defined by $\varphi(y) := \inf\{z \in [0, \infty] : \psi(z) = y\}$. We call an Archimedean generator ψ strict if $\varphi(0) = \infty$ (or, equivalently, if $\psi(x) > 0$) for every $x \in (0, \infty)$) and non-strict otherwise. A copula $A \in \mathcal{C}^d$ is called Archimedean if there exists an Archimedean generator ψ such that

$$A(\mathbf{x}) = \psi\left(\varphi(x_1) + \varphi(x_2) + \ldots + \varphi(x_d)\right) \tag{2.1}$$

holds for all $\mathbf{x} \in \mathbb{I}^d$. In McNeil and Neslehová (2009) the authors established the following equivalence: $A_\psi \in \mathcal{C}^d$ according to eq. (2.1) is a d-dimensional Archimedean copula if, and only if, ψ is a d-monotone Archimedean generator on $[0, \infty)$; thereby ψ is called d-monotone if $(-1)^{d-2}\psi^{(d-2)}$ exists, is non-negative, non-increasing and convex on $(0, \infty)$. We will write A_ψ for the Archimedean copula with generator ψ and let $\mathcal{C}^d_{ar}$ denote the family of all d-dimensional Archimedean copulas.

In the sequel we work with level sets of copulas and the Kendall distribution function. For every function $f : \mathbb{I}^m \to \mathbb{I}$ (with $m \geq 1$) the upper/lower t-cut is defined by

$$[f]_t := \{\mathbf{x} \in \mathbb{I}^m : f(\mathbf{x}) \geq t\}, \quad \langle f\rangle_t := \{\mathbf{x} \in \mathbb{I}^m : f(\mathbf{x}) \leq t\}.$$

Using this notation, the Kendall distribution function $F_C^{Kendall} : \mathbb{I} \to \mathbb{I}$ of $C \in \mathcal{C}^d$ is given by $F_C^{Kendall}(t) = \mu_C(\langle C\rangle_t)$ for every $t \in \mathbb{I}$. In what follows we will let κ_C denote the probability measure corresponding to $F_C^{Kendall}$. Following Kasper et al. (2024), for $A_\psi \in \mathcal{C}^d_{ar}$ and every interval $I \subseteq \mathbb{I}$ we will write

$$L_I := \left\{(\mathbf{x}, y) \in \mathbb{I}^d : A_\psi(\mathbf{x}, y) \in I\right\}. \tag{2.2}$$

For the special case $I = \{t\}$ we simply write L_t instead of $L_{\{t\}}$ and refer to L_t as t-level set of $A_\psi \in \mathcal{C}^d_{ar}$. Obviously we have

$$L_t := [A_\psi]_t \cap \langle A_\psi \rangle_t = \left\{ (\mathbf{x}, y) \in \mathbb{I}^{d-1} \times \mathbb{I} : \sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y) = \varphi(t) \right\} \tag{2.3}$$

as well as

$$L_0 := \left\{ (\mathbf{x}, y) \in \mathbb{I}^{d-1} \times \mathbb{I} : \sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y) \geq \varphi(0) \right\}. \tag{2.4}$$

To keep notation simple, $L_t^{1:d-1}$ will denote the t-level set of the marginal copula $A_\psi^{1:d-1}$. The so-called t-level function $f^t : \left[A_\psi^{1:d-1}\right]_t \to \mathbb{I}$ is defined by

$$f^t(\mathbf{x}) = \psi \left(\varphi(t) - \sum_{i=1}^{d-1} \varphi(x_i) \right) \tag{2.5}$$

for $t \in (0, 1]$, and for $t = 0$ by

$$f^0(\mathbf{x}) = \psi \left(\varphi(0) - \sum_{i=1}^{d-1} \varphi(x_i) \right) \tag{2.6}$$

for every $\mathbf{x} \notin L_0^{1:d-1}$, whereby we set $\psi(x) = 1$ for all $x < 0$. Obviously for $t \in (0, 1]$ the graph of the t-level function and the level set L_t coincide, i.e., $\Gamma(f^t) := \{(\mathbf{x}, f^t(\mathbf{x})) : \mathbf{x} \in \left[A_\psi^{1:d-1}\right]_t\} = L_t$ holds. Following Kasper et al. (2024) (a version of) the Markov kernel K_{A_ψ} of $A_\psi \in \mathcal{C}^d$ is given by (D^- denoting the left-hand derivative)

$$K_{A_\psi}(\mathbf{x}, [0, y]) := \begin{cases} 1, & M(\mathbf{x}) = 1 \text{ or } \mathbf{x} \in L_0^{1:d-1} \\ 0, & M(\mathbf{x}) < 1, \mathbf{x} \notin L_0^{1:d-1}, y < f^0(\mathbf{x}) \\ \frac{D^-\psi^{(d-2)}\left(\sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y)\right)}{D^-\psi^{(d-2)}\left(\sum_{i=1}^{d-1} \varphi(x_i)\right)} & M(\mathbf{x}) < 1, \mathbf{x} \notin L_0^{1:d-1}, y \geq f^0(\mathbf{x}). \end{cases} \tag{2.7}$$

For establishing our main results we will make use of the interplay between the so-called Williamson measures and Archimedean copulas and follow McNeil and Neslehová (2009); Kasper et al. (2024); Dietrich and Trutschnig (2025). According to McNeil and Neslehová (2009) it can be shown that ψ is the Archimedean generator of a d-dimensional Archimedean copula A_ψ if, and only if, there exists some unique probability measure γ on $\mathcal{B}([0, \infty))$ with $\gamma(\{0\}) = 0$ such that

$$\psi(z) = \int_{\left[0, \frac{1}{z}\right)} (1 - tz)^{d-1} \, d\gamma(t) \tag{2.8}$$

holds for every $z > 0$. The measure γ is referred to as Williamson measure. According to Kasper et al. (2024) eq. (2.8) implies the following explicit expression for $D^-\psi^{(d-2)}$ in terms of γ:

$$0 \geq G(z) := (-1)^{d-2} D^- \psi^{(d-2)}(z) = -(d-1)! \int\limits_{(0,\frac{1}{z}]} t^{d-1}\, d\gamma(t), \quad z > 0. \tag{2.9}$$

According to Kasper et al. (2024) the Kendall distribution function $F_{A_\psi}^{Kendall}$ associated with A_ψ can be expressed as

$$F_{A_\psi}^{Kendall}(t) = \gamma\left(\left[0, \tfrac{1}{\varphi(t)}\right]\right), \quad t \in (0,1]. \tag{2.10}$$

Following Trutschnig and Fernández Sánchez (2025), considering $r \in (0, \frac{1}{2})$, defining the contractions $w_1^r, w_2^r : [0,1] \to [0,1]$ by $w_1^r(x) = rx$, $w_2^r(x) = 1 - rx$, and setting $p_1 = p_2 = \frac{1}{2}$ yields the totally disconnected Iterated Function System with probabilities (IFSP) $\{[0,1], (w_i^r)_{i=1}^2, (p_i)_{i=1}^2\}$. The induced Hutchinson and Markov operator, defined by $\mathcal{H}_r(E) = w_1^r(E) \cup w_2^r(E)$ and $\mathcal{V}_r(\vartheta) = p_1 \vartheta^{w_1} + p_2 \vartheta^{w_2}$ map $\mathcal{K}(\mathbb{I}^2)$ into $\mathcal{K}(\mathbb{I}^2)$ and $\mathcal{P}(\mathbb{I}^2)$ into $\mathcal{P}(\mathbb{I}^2)$, respectively, and have unique (globally attractive) fixed points $Z_r^* \in \mathcal{K}([0,1])$ and $\vartheta_r^* \in \mathcal{P}([0,1])$ (see Kunze et al. (2012)). Moreover, these two fixed points are connected via

$$\operatorname{supp}(\vartheta_r^*) = Z_r^*. \tag{2.11}$$

Notice that each Z_r^* is a Cantor set (Z_r^* for $r = \frac{1}{3}$, e.g., is the well-known middle-thirds Cantor set) and as such a perfect set, in particular each Z_r^* coincides with its limit points. Setting

$$\begin{aligned} &J_1 = (r, 1-r), \quad J_2 = w_1^r(J_1), \quad J_3 = w_2^r(J_1), \quad J_4 = w_1^r \circ w_1^r(J_1), \\ &J_5 = w_1^r \circ w_2^r(J_1), \quad J_6 = w_2^r \circ w_1^r(J_1), \quad J_7 = w_2^r \circ w_2^r(J_1), \quad \ldots \end{aligned}$$

it is straightforward to verify that the intervals are pairwise disjoint, open and fulfill $\mathbb{I} \backslash Z_r^* = \bigcup_{k=1}^\infty J_k$.

3. Archimedean copulas with fractal support in dimension $d = 3$

For what follows let $r \in (0, \frac{1}{2})$ be arbitrary but fixed. In a nutshell, our construction of a three-dimensional Archimedean copula with fractal support builds upon the generator ψ_r studied in the two-dimensional setting in Trutschnig and Fernández Sánchez (2025) and considers the normalized anti-derivative ψ of ψ_r as well as the Archimedean copula A_ψ.

Let G_r^* denote the Cantor-like distribution function corresponding to the measure ϑ_r^* and define the (strictly decreasing and convex) function $\varphi_r : \mathbb{I} \to \mathbb{R}$ by

$$\varphi_r(x) := \int_{[0,x]} -2 + 2\, G_r^*(t)\, d\lambda(t) + 1.$$

Then φ_r is a homeomorphism and the corresponding non-strict Archimedean generator $\psi_r : [0, \infty) \to \mathbb{I}$ is given by $\psi_r(t) = \varphi_r^{-1}(t)\mathbf{1}_{\mathbb{I}}(t)$ (again see Trutschnig and Fernández Sánchez (2025)). Based on ψ_r define a (non-strict) Archimedean generator $\psi : [0, \infty) \to \mathbb{I}$ by

$$\psi(s) := \tfrac{1}{C} \int_{[s,1]} \psi_r(u)\, d\lambda(u), \tag{3.1}$$

with $C := \int_{[0,1]} \psi_r(u)\, d\lambda(u) > 0$ serving as a normalizing constant assuring $\psi(0) = 1$. It is straightforward to see that ψ is 3-monotone since $(-1)\psi' = \frac{1}{C}\psi_r$ is non-increasing, non-negative and convex on $(0, \infty)$. We denote the pseudo-inverse of ψ by φ and, simplifying notation, write $\psi'(0) := D^+\psi(0)$ (with D^+ denoting the right-hand derivative) and $\psi'(1) := D^-\psi(1)$.
Given an arbitrary function $h : [0, 1] \to \mathbb{R}$ we will refer to the set

$$\mathcal{M}(h) := \{x \in (0, 1) : |h(x + \delta) - h(x - \delta)| > 0 \text{ for all } 0 < \delta < \min\{x, 1 - x\}\}$$

as the set of points of strict monotonicity of h. Notice that in case of h being a distribution function (or, more generally, a measure-generating function) $\mathcal{M}(h)$ coincides with the support of the measure induced by h (intersected with $(0, 1)$).

According to eq. (2.7) for $t \in (0, 1)$ and $x \in (t, 1)$ we have

$$K_{A_{\psi_r}}(x, [0, \psi_r(\varphi_r(t) - \varphi_r(x))]) = \tfrac{\psi_r'(\varphi_r(t))}{\psi_r'(\varphi_r(x))}, \tag{3.2}$$

so $y \mapsto K_{A_{\psi_r}}(x, [0, y])$ has a point of strict monotonicity in $\psi_r(\varphi_r(t) - \varphi_r(x))$ if, and only if, $\varphi_r(t)$ is a point of strict monotonicity of ψ_r'. Using the fact that $\varphi_r'(z) = (\psi_r^{-1}(z))' = \frac{1}{\psi_r'(\varphi_r(z))}$ holds on $(0, 1)$, it follows that

$$\psi_r'(z) = \tfrac{1}{\varphi_r'(\psi_r(z))} = \tfrac{1}{-2+2G_r^*(\psi_r(z))}, \quad z \in (0, 1). \tag{3.3}$$

As a direct consequence, for every $t \in (0, 1)$ we have $\varphi_r(t) \in \mathcal{M}(\psi_r')$ if, and only if, $t \in \mathcal{M}(G_r^*) = Z_r^* \cap (0, 1)$. In other words, for $t \in (0, 1)$ the following equivalence holds:

$$t \in Z_r^* \iff \psi_r(\varphi_r(t) - \varphi_r(x)) \in \operatorname{supp}(K_{A_{\psi_r}}(x, \cdot)) \text{ for every } x \in (t, 1) \tag{3.4}$$

We now tackle the three-dimensional setting, start by proving the corresponding version of equivalence (3.4), and determining $\dim_H(\psi \circ \varphi_r(Z_r^*))$.

Lemma 3.1. *For the Archimedean copula $A_\psi \in \mathcal{C}^3_{ar}$ with ψ according to eq. (3.1) and every $t \in (0,1)$ the following two assertions are equivalent:*

1. $t \in \psi \circ \varphi_r(Z_r^*)$,
2. $f^t(\mathbf{x}) \in supp(K_{A_\psi}(\mathbf{x}, \cdot))$ *for every* $\mathbf{x} \in int(\left[A_\psi^{1:2}\right]_t)$.

Proof. Considering $\psi'' = -\frac{1}{C}\psi_r'$ we obviously have $\mathcal{M}(\psi'') = \mathcal{M}(-\frac{1}{C}\psi_r') = \mathcal{M}(\psi_r') = \varphi_r(Z_r^*)$. For $\mathbf{x} = (x_1, x_2) \in int(\left[A_\psi^{1:2}\right]_t)$ and arbitrary $y \in (f^0(\mathbf{x}), 1]$ we have

$$K_{A_\psi}(\mathbf{x}, [0,y]) = \tfrac{\psi''(\varphi(x_1)+\varphi(x_2)+\varphi(y))}{\psi''(\varphi(x_1)+\varphi(x_2))} = \tfrac{\psi_r'(\varphi(x_1)+\varphi(x_2)+\varphi(y))}{\psi_r'(\varphi(x_1)+\varphi(x_2))},$$

which, for the special case of $y = f^t(\mathbf{x}) \in (0,1)$ boils down to

$$K_{A_\psi}(\mathbf{x}, [0, f^t(\mathbf{x})]) = \tfrac{\psi_r'(\varphi(t))}{\psi_r'(\varphi(x_1)+\varphi(x_2))}.$$

Having this directly yields that for $\mathbf{x} \in int(\left[A_\psi^{1:2}\right]_t)$ we have $f^t(\mathbf{x}) \in \mathrm{supp}(K_{A_\psi}(\mathbf{x}, \cdot))$ if, and only if, $\varphi(t) \in \mathcal{M}(\psi_r') = \varphi_r(Z_r^*)$, which, considering that φ is a homeomorphism of $\mathbb{I}$ completes the proof. □

Lemma 3.2. *The Hausdorff dimension of the set $\psi \circ \varphi_r(Z_r^*)$ coincides with the Hausdorff dimension of Z_r^*, i.e., we have*

$$\dim_H(\psi \circ \varphi_r(Z_r^*)) = -\tfrac{\log(2)}{\log(r)}. \tag{3.5}$$

Proof. It is well known that $\dim_H(Z_r^*) = -\frac{\log(2)}{\log(r)}$ holds. Using invariance of the Hausdorff dimension under bi-Lipschitz transformations and countable stability of the Hausdorff dimension (see, e.g., Falconer (2003)) we proceed as follows (notice that $\psi \circ \varphi_r$ is not bi-Lipschitz on $\mathbb{I}$): For every $n \in \mathbb{N}$ consider the compact interval

$$E_n := \left[\tfrac{1}{2}r^n, 1 - \tfrac{1}{2}r^n\right] \subset (0,1). \tag{3.6}$$

Obviously φ_r, ψ_r, ψ and φ are bi-Lipschitz on every compact subinterval of $(0,1)$. Hence, considering that compositions of bi-Lipschitz transformations are bi-Lipschitz too, it follows that $\psi \circ \varphi_r$ is bi-Lipschitz on E_n for every $n \in \mathbb{N}$. Moreover, since the intervals E_n (due to self-similarity of Z_r^*) fulfil $\dim_H(Z_r^* \cap E_n) = \dim_H(Z_r^*)$ and since single points have Hausdorff dimension 0, using countable stability of the Hausdorff-dimension directly yields the asserted identity via

$$\begin{aligned}\dim_H(\psi \circ \varphi_r(Z_r^*)) &= \dim_H(\psi \circ \varphi_r(Z_r^*) \cap (0,1)) = \dim_H\left(\bigcup_{n\in\mathbb{N}} \psi \circ \varphi_r(E_n \cap Z_r^*)\right) \\ &= \sup_{n\in\mathbb{N}} \dim_H\left(\psi \circ \varphi_r(E_n \cap Z_r^*)\right) = \dim_H(Z_r^*).\end{aligned}$$

□

To simplify notation for every $k \in \mathbb{N}$ define the open interval I_k by

$$I_k = \psi \circ \varphi_r(J_k). \tag{3.7}$$

Then obviously $(I_k)_{k\in\mathbb{N}}$ fulfills $\mathbb{I}\backslash(\psi \circ \varphi_r(Z_r^*)) = \bigcup_{k=1}^{\infty} I_k$ and that the intervals are pairwise disjoint.

Lemma 3.3. *The following identity holds:*

$$\mu_{A_\psi}\left(\bigcup_{k=1}^{\infty} L_{I_k}\right) = 0 = \kappa_{A_\psi}\left(\bigcup_{k=1}^{\infty} I_k\right) \tag{3.8}$$

Proof. For dimension $d = 3$ applying eq. (2.9) yields

$$-2 \int_{\left(0,\frac{1}{z}\right]} t^2 \, d\gamma(t) = -\psi''(z) = \tfrac{1}{C}\psi_r'(z) = \tfrac{1}{C}\tfrac{1}{-2+2G_r^*(\psi_r(z))} \tag{3.9}$$

for every $z \in (0, 1)$. Now, since G_r^* is constant on each J_k and the integrand t^2 is strictly positive, eq. (3.9) implies that the Williamson measure γ assigns no mass to the open interval $\frac{1}{\varphi_r(J_k)} = \left\{\frac{1}{\varphi_r(s)} : s \in J_k\right\}$ for every $k \geq 1$. Using eq. (2.10) it follows that the Kendall distribution function $F_{A_\psi}^{Kendall}$ is constant on the interval $\psi \circ \varphi_r(J_k) = I_k$, which shows the second equality in eq. (3.8). As a direct consequence, considering $I_k =: (a, b)$, the definition of the Kendall distribution function implies

$$\mu_{A_\psi}(L_{(a,b)}) = F_{A_\psi}^{Kendall}(b-) - F_{A_\psi}^{Kendall}(a) = F_{A_\psi}^{Kendall}(b) - F_{A_\psi}^{Kendall}(a) = 0.$$

Since $k \in \mathbb{N}$ was arbitrary this completes the proof. □

Building upon the previous lemma we can now state and prove the first main result.

Lemma 3.4. *The support supp(μ_{A_ψ}) of μ_{A_ψ} is given by*

$$supp(\mu_{A_\psi}) = \bigcup_{t \in \psi \circ \varphi_r(Z_r^*)} \Gamma(f^t). \tag{3.10}$$

Furthermore, the support of the probability measure κ_{A_ψ} is the set $\psi \circ \varphi_r(Z_r^*)$.

Proof. (a) We first show that for every $t \in \psi \circ \varphi_r(Z_r^*) \cap (0, 1)$ and every $\mathbf{x} \in int\left(\left[A_\psi^{1:2}\right]_t\right)$ the point $(\mathbf{x}, f^t(\mathbf{x}))$ is an element of the support of μ_{A_ψ}. We already know from Lemma 3.1 that with t and $\mathbf{x}$ as above

$$f^t(\mathbf{x}) \in \text{supp}(K_{A_\psi}(\mathbf{x}, \cdot))$$

holds. Choose $\delta > 0$ sufficiently small such that

$$Q := (x_1 - \delta, x_1 + \delta) \times (x_2 - \delta, x_2 + \delta) \times (f^t(\mathbf{x}) - \delta, f^t(\mathbf{x}) + \delta) \subseteq int\left(\left[A_\psi^{1:2}\right]_t\right) \times (0,1)$$

holds. Using continuity of the function f^t on $\left[A_\psi^{1:2}\right]_t$ we can find some set non-degenerated open set $U \subseteq (x_1 - \delta, x_1 + \delta) \times (x_2 - \delta, x_2 + \delta) =: V$ such that $f^t(s_1, s_2) \in (f^t(\mathbf{x}) - \delta, f^t(\mathbf{x}) + \delta)$ for every $(s_1, s_2) \in U$. Applying disintegration, we obtain

$$\begin{aligned}
\mu_{A_\psi}(Q) &= \int_V K_{A_\psi}\left(\mathbf{s}, (f^t(\mathbf{x}) - \delta, f^t(\mathbf{x}) + \delta)\right) d\mu_{A_\psi^{1:2}}(\mathbf{s}) \\
&\geq \int_U \underbrace{K_{A_\psi}\left(\mathbf{s}, (f^t(\mathbf{x}) - \delta, f^t(\mathbf{x}) + \delta)\right) d\mu_{A_\psi^{1:2}}(\mathbf{s})}_{>0} > 0,
\end{aligned}$$

whereby the last (strict) inequality results from the following observation: According to McNeil and Neslehová (2009) the marginal copula $A_\psi^{1:2}$ is absolutely continuous and a version $k_{A_\psi^{1:2}}$ of its density is strictly positive on the set

$$\{(x_1, x_2) \in (0,1)^2 : x_2 \geq \psi(\varphi(0) - \varphi(x_1))\}. \tag{3.11}$$

This implies $\mu_{A_\psi^{1:2}}(U) > 0$. Considering that $\delta > 0$ was arbitrary, we have shown that μ_{A_ψ} assigns positive mass to any non-degenerated open cube around $(\mathbf{x}, f^t(\mathbf{x}))$, which directly yields $(\mathbf{x}, f^t(\mathbf{x})) \in \operatorname{supp}(\mu_{A_\psi})$.
(b) Since $\operatorname{supp}(\mu_{A_\psi})$ is compact (and hence closed), continuity of the mapping f^t implies

$$\operatorname{supp}(\mu_{A_\psi}) \supseteq \bigcup_{t \in \psi \circ \varphi_r(Z_r^*) \cap (0,1)} \Gamma(f^t).$$

Moreover, since $Z_r^\star$ coincides with its limit points the same holds true for $\psi \circ \varphi_r(Z_r^\star)$. As a direct consequence, there exists a strictly decreasing sequence $(t_n)_{n \in \mathbb{N}}$ of points in $\psi \circ \varphi_r(Z_r^\star)$ with limit 0 and for every $\mathbf{x} \notin L_0^{1:2}$ we can find a sequence $(\mathbf{x}_n, f^{t_n}(\mathbf{x}_n))_{n \in \mathbb{N}}$ converging to $(\mathbf{x}, f^0(\mathbf{x}))$ with the following property:

$$(\mathbf{x}_n, f^{t_n}(\mathbf{x}_n)) \in int\left(\left[A_\psi^{1:2}\right]_{t_n}\right) \times (0,1) \quad \text{for every } n \in \mathbb{N}.$$

Again using the fact that $\operatorname{supp}(\mu_{A_\psi})$ is closed we conclude that $(\mathbf{x}, f^0(\mathbf{x})) \in \operatorname{supp}(\mu_{A_\psi})$. Proceeding analogously for f^1 altogether we have shown

$$\operatorname{supp}(\mu_{A_\psi}) \supseteq \bigcup_{t \in \psi \circ \varphi_r(Z_r^*)} \Gamma(f^t).$$

Considering that Lemma 3.3 directly yields

$$\operatorname{supp}(\mu_{A_\psi}) \subseteq [0,1]^2 \setminus \bigcup_{k=1}^{\infty} L_{I_k} = \bigcup_{t\in\psi\circ\varphi_r(Z_r^*)} \Gamma(f^t),$$

and hence completes the proof of eq. (3.10). The second assertion is a consequence of the definition of the Kendall distribution function, Lemma 3.3, and eq. (3.10). □

Having collected all auxiliary results we can now proceed with proving the main theorem of this paper (see left panel of Figure 1 for an approximation of the support):

Theorem 3.5. *The Hausdorff dimension of the support of μ_{A_ψ} is given by*

$$\dim_H(supp(\mu_{A_\psi})) = 2 - \tfrac{\log(2)}{\log(r)}. \tag{3.12}$$

Proof. We again use the fact that bi-Lipschitz transformations preserve the Hausdorff dimension and proceed in three steps.
(i) For every $t \in [0,1]$ let T_t denote the triangle with vertices $(t,t),(1,t),(1,1)$ and define the pyramid Δ and the compact set Λ by

$$\Delta := \bigcup_{t\in[0,1]} T_t \times \{t\}, \quad \Lambda = \bigcup_{t\in\psi\circ\varphi_r(Z_r^*)} T_t \times \{t\}.$$

For each $t \in [0,1]$, letting $h_t : T_t \to [0,1]^2$ denote the transformation defined by $h_t(x,s) := (x, \psi(\varphi(s)-\varphi(x)))$, we have that h_t maps T_t bijectively to $[A_\psi^{1:2}]_t$. More importantly, the transformation $H : \Delta \to [0,1]^3$, defined by

$$H(x,s,t) := \big(h_t(x,s), f^t(h_t(x,s))\big) = (x, \psi(\varphi(s)-\varphi(x)), \psi(\varphi(t)-\varphi(s)))\,,$$

maps each $T_t \times \{t\}$ to $\Gamma(f^t)$. It is straightforward to verify that H is bijective on Δ and the construction implies that we have $H(\Lambda) = \operatorname{supp}(\mu_{A_\psi})$.
(ii) Consider $E_n = [\frac{r^n}{2}, 1-\frac{r^n}{2}]$, write $\psi\circ\varphi_r(E_n) =: [l_n, u_n] \subset (0,1)$, and set $\delta_n = \frac{1-u_n}{8}$. We approximate each triangle T_t (from within) by triangles $T_{t,n}$, given by

$$T_{t,n} = \operatorname{convh}(\{(t+2\delta_n, t+\delta_n),\, (1-\delta_n, t+\delta_n),\, (1-\delta_n, 1-2\delta_n)\}) \subset \operatorname{int}(T_t)$$

for $n \in \mathbb{N}$, and set

$$\Delta_n = \bigcup_{t\in[l_n,u_n]} T_{t,n} \times \{t\}, \quad \Lambda_n = \bigcup_{t\in\psi\circ\varphi_r(Z_r^*\cap E_n)} T_{t,n} \times \{t\}.$$

The affine transformation $g_n : \Delta_n \to \Delta_n$, defined by

$$g_n(x_1,x_2,t) = (1-\delta_n + (\tfrac{1-l_n-3\delta_n}{1-t-3\delta_n})(x_1-(1-\delta_n)), 1-2\delta_n + (\tfrac{1-l_n-3\delta_n}{1-t-3\delta_n})(x_2-(1-2\delta_n)), t),$$

maps $T_{t,n} \times \{t\}$ to $T_{l_n,n} \times \{t\}$ and has $(1-\delta_n, 1-2\delta_n, t)$ as a fixed point. Moreover, considering that $g_n : \Delta_n \to \Delta_n$ is bi-Lipschitz, it follows that

$$\dim_H(\Lambda_n) = \dim_H(g_n(\Lambda_n)) = \dim_H\left(T_{l_n,n} \times \psi \circ \varphi_r(Z_r^* \cap E_n)\right) = 2 - \tfrac{\log(2)}{\log(r)},$$

where the last equality is a direct consequence of the Marstrand Product Theorem for the Hausdorff dimension (see Bishop and Peres (2016)).
(iii) From (ii), together with countable stability of the Hausdorff dimension and the fact that H is bi-Lipschitz on each Λ_n, we obtain

$$\begin{aligned}\dim_H \underbrace{\left(H\left(\bigcup_{n\geq 2} \Lambda_n\right)\right)}_{=:\Upsilon} &= \dim_H\left(\bigcup_{n\geq 2} H(\Lambda_n)\right) = \sup_{n\geq 2} \dim_H(H(\Lambda_n)) \\ &= \sup_{n\geq 2} \dim_H(\Lambda_n) = 2 - \tfrac{\log(2)}{\log(r)}.\end{aligned}$$

Since the support of μ_{A_ψ} is the union of the sets Υ, $\Gamma(f^0)$ and the set $U := \text{supp}(\mu_{A_\psi}) \cap ([0,1]^2 \times \{1\})$ and since the latter two sets have a Hausdorff dimension of at most $2 < 2 - \frac{\log(2)}{\log(r)}$ it follows immediately that $\dim_H(\text{supp}(\mu_{A_\psi})) = \dim_H(\Lambda) = 2 - \frac{\log(2)}{\log(r)}$. □

Summing up, since $r \in (0, \frac{1}{2})$ was arbitrary we have proved the following result:

Theorem 3.6. *For every $s \in (2,3)$ there exists some Archimedean copula $A_\psi \in \mathcal{C}_{ar}^3$ with the following properties:*

1. $\dim_H(\mathit{supp}(\mu_{A_\psi})) = s$
2. $\dim_H(\mathit{supp}(\kappa_{A_\psi})) = s - 2$

Proof. Considering that the mapping $\iota : (0, \frac{1}{2}) \to (0,1)$, defined by $\iota(r) = -\frac{\log(2)}{\log(r)}$, is surjective, combining Lemma 3.2, Lemma 3.4 and Theorem 3.5 directly yields the desired result. □

Remark 3.7. While the Archimedian copula A_ψ in the proof of Theorem 3.6 has a pathological support, the bivariate marginal copulas are much more regular: In fact, considering eq. (3.11) or working directly with the Markov kernel of $A_\psi^{1:2}$, given by

$$K_{A_\psi^{1:2}}(x_1, [0, x_2]) = \tfrac{\psi'(\varphi(x_1)+\varphi(x_2))}{\psi'(\varphi(x_1))} = \tfrac{\psi_r(\varphi(x_1)+\varphi(x_2))}{\psi_r(\varphi(x_1))},$$

and noticing that for every $x_1 \in (0,1)$ the mapping $x_2 \mapsto K_{A_\psi^{1:2}}(x_1, [0, x_2])$ is strictly increasing on the interval $[\psi(\varphi(0) - \varphi(x_1)), 1]$, $\dim_H(\text{supp}(\mu_{A_\psi^{1:2}})) = 2$ follows immediately. The right panel in Figure 1 depicts a sample of the marginal copula $A_\psi^{1:2}$.

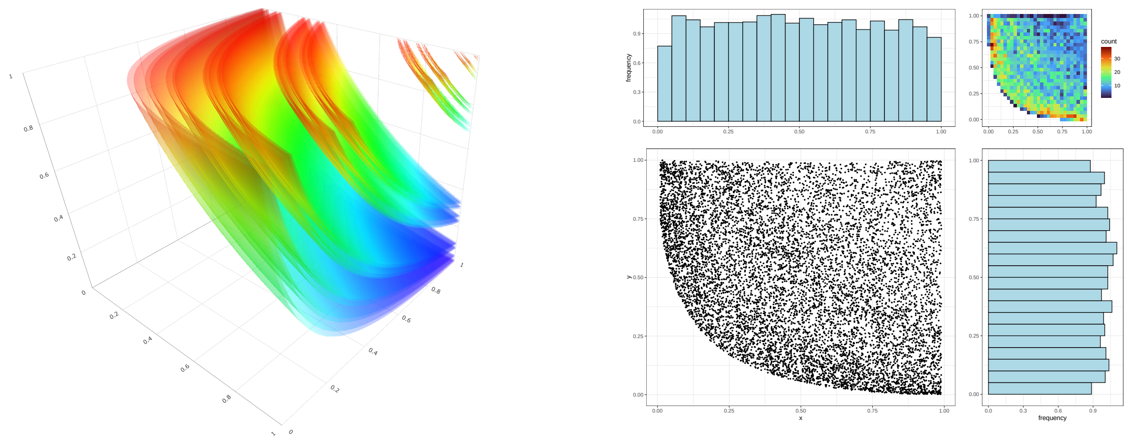

Figure 1: (a) Approximation of the support of A_ψ for the situation $r = \frac{1}{3}$ (left plot). (b) Sample of size $n = 10.000$ of the absolutely continuous marginal copula $A_\psi^{1:2}$; two-dimensional histogram (upper right panel) and marginal histograms (upper left and lower right panels).

Starting with eq. (3.1) and advancing in the same manner as in the three-dimension setting allows to derive the analogous result for dimension $d = 4$. Proceeding in this stepwise manner eventually we obtain the following results for arbitrary dimension $d \geq 3$:

Theorem 3.8. *For every $d \geq 3$ and arbitrary $s \in (d-1, d)$ there exists some Archimedean copula whose support has Hausdorff dimension s.*

4. Conclusion and outlook

Our paper underlines the fact that even in the supposedly regular/smooth family of d-dimensional Archimedean copulas, very irregular mass distributions are possible. More importantly, our construction shows that ignoring (at least) one coordinate may create a false impression of regularity: in fact, considering $\mathbf{X} = (X_1, \ldots, X_d) \sim A_\psi$ with $\dim_H(\mathrm{supp}(\mu_{A_\psi})) = s \in (d-1, d)$, we have that all marginal copulas of A_ψ (i.e., all copulas of subvectors of $\mathbf{X}$) are absolutely continuous and exhibit a very regular support. Moving away from the Archimedean family, one natural question is, whether analogous constructions for d-dimensional copulas distributing mass in a pathological way despite having very regular marginals are possible in other classes. Moreover, focusing on dependence in general and moving from multivariate statistics to nonparametric functional data analysis (NPFDA, see Ling and Vieu (2018) for a recent survey), one line of our future work will be to analyze (the potential of) analogous constructions in the context of functional PCA. Interestingly, as shown in Ferraty et al. (2002), the asymptotics of a Nadaraya-Watson type nonparametric estimator for the regression function are linked with the fractal dimension of the underlying functional process.

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