How far are d-dimensional copulas with uniform $(d-1)$-marginals from (total) independence?

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Abstract

We consider the family $\mathcal{C}_d^{\Pi_{d-1}}$ of all d-dimensional copulas whose $(d-1)$-dimensional marginals are all equal to the $(d-1)$-dimensional product copula Π_{d-1} and tackle the natural question, 'how far away' from the d-dimensional product copula Π_d elements in $\mathcal{C}_d^{\Pi_{d-1}}$ can be. We provide definitive answers for both, the uniform metric d_∞ as well as the stronger, conditioning-based metric D_1. The established results clearly indicate that the family $\mathcal{C}_d^{\Pi_{d-1}}$ is larger than one might expect.

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1. Introduction

In the paper by Nelsen and Úbeda-Flores (2012), entitled 'How close are pairwise and mutual independence?' the authors derived sharp pointwise bounds for the family of all three-dimensional copulas with uniform bivariate marginals and provided an asymptotic result for the general d-dimensional

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setting with $d \geqslant 3$. Moving from bivariate uniform to $(d-1)$-dimensional uniform marginals, we here consider $d \geqslant 3$ and study the class $\mathcal{C}_d^{\Pi_{d-1}}$ of all d-dimensional copulas whose $(d-1)$-dimensional marginals all coincide with the $(d-1)$-dimensional product copula Π_{d-1}. The intuitive conjecture, formulated in Nelsen and Úbeda-Flores (2012), saying that pairwise independence should restrict variability/flexibility as well as the maximum possible distance to mutual (total) independence modelled by Π_d, naturally applies even more for the family $\mathcal{C}_d^{\Pi_{d-1}}$. As our results show, however, $\mathcal{C}_d^{\Pi_{d-1}}$ is much larger and versatile than one might expect – regardless of whether we consider the standard uniform metric d_∞ or the stronger metric D_1 (originally introduced in Trutschnig (2011) in the bivariate setting and generalized in Griessenberger et al. (2022) to the multivariate case to quantify the extent of dependence of a random variable Y on a random vector $\mathbf{X}$). As our main results we will prove that

$$\sup\left\{d_\infty(A,\Pi_d):\ A\in\mathcal{C}_d^{\Pi_{d-1}}\right\}=\tfrac{1}{2^d}$$

as well as

$$\sup\left\{D_1(A,\Pi_d):\ A\in\mathcal{C}_d^{\Pi_{d-1}}\right\}=\tfrac{1}{3}$$

holds for every $d \geqslant 3$ and provide examples showing that both suprema are attained.

The remainder of this note is organized as follows: Section 2 gathers notation and preliminaries, Section 3 presents and proves the main results. Several examples and figures illustrate the underlying ideas.

2. Notation and Preliminaries

Throughout the paper bold symbols will denote vectors. For every fixed index/coordinate $i_0 \in [d] := \{1,\dots,d\}$ and every $\mathbf{x} \in \mathbb{R}^d$ we will write $\mathbf{x}_{-i_0} = (x_1,\dots,x_{i_0-1},x_{i_0+1},\dots,x_d)$. Moreover, for every vector $\mathbf{j} = (j_1,\dots,j_l)$ of at most $l \leqslant d-1$ pairwise distinct indices in $[d]$ we will denote by $\pi_{\mathbf{j}} : [0,1]^d \to [0,1]^l$ the projection onto the coordinates in $\mathbf{j}$, i.e., $\pi_{\mathbf{j}}(x_1,\dots,x_d) = (x_{j_1},\dots,x_{j_l})$. To simplify notation we will also write $\pi_{1:l}$ for the projection onto the first l coordinates as well as π_{-j_0} for the projection onto the coordinates $(1,\dots,j_0-1,j_0+1,\dots,d)$.

As already mentioned before, $\mathcal{C}_d$ will denote the class of all d-dimensional copulas for $d \geqslant 2$, and Π_d the d-dimensional product copula. We will write $\mathcal{P}_{\mathcal{C}_d}$ for the class of all d-stochastic measures (i.e., probability measures for which the corresponding distribution function restricted to $[0,1]^d$

is a d-dimensional copula). For every $A \in \mathcal{C}_d$ the corresponding d-stochastic measure will be denoted by $\mu_A \in \mathcal{P}_{\mathcal{C}_d}$; moreover, $\lambda_d = \mu_{\Pi_d}$ will denote the d-dimensional Lebesgue measure on $[0,1]^d$. For a random vector $\mathbf{X} = (X_1, \ldots, X_d)$ on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $A \in \mathcal{C}_d$ we will write $\mathbf{X} \sim A$ if, and only if, A is the restriction of the distribution function of $\mathbf{X}$ to $[0,1]^d$ (or, equivalently, if the distribution $\mathbb{P}^{\mathbf{X}}$ of $\mathbf{X}$ coincides with μ_A). Notice that for $\mathbf{X} = (X_1, \ldots, X_d) \sim A$ we have that each X_i is uniformly distributed on $[0,1]$.

For every $A \in \mathcal{C}_d$ and every vector $\mathbf{j} = (j_1, \ldots, j_l)$ of $l \leqslant d-1$ distinct indices in $[d]$ we will let $A^{\mathbf{j}}$ denote the marginal copula corresponding to $\mathbf{j}$, i.e., the copula corresponding to the push-forward $\mu_{A^{\mathbf{j}}} = \mu_A^{\pi_{\mathbf{j}}}$ of μ_A via $\pi_{\mathbf{j}}$; $A^{1:l}$ denotes the marginal copula of the first l coordinates and A^{-j_0} the marginal copula of all coordinates except coordinate j_0. For further properties of copulas and d-stochastic measures we refer the reader to Durante and Sempi (2016), and Nelsen (2006).

For a topological space T we will denote the Borel σ-algebra on T by $\mathcal{B}(T)$. We call a map $K : \mathbb{R}^{d-1} \times \mathcal{B}(\mathbb{R}) \to [0,1]$ a $(d-1)$-*Markov kernel* from $\mathbb{R}^{d-1}$ to $\mathbb{R}$, if the function $\mathbf{x} \mapsto K(\mathbf{x}, E)$ is $\mathcal{B}(\mathbb{R}^{d-1})$-$\mathcal{B}(\mathbb{R})$-measurable for every fixed $E \in \mathcal{B}(\mathbb{R})$ and the map $E \mapsto K(\mathbf{x}, E)$ is a probability measure on $\mathcal{B}(\mathbb{R})$ for every $\mathbf{x} \in \mathbb{R}^{d-1}$. Given a random variable Y and a $(d-1)$-dimensional random vector $\mathbf{X}$ on a joint probability space $(\Omega, \mathcal{A}, \mathbb{P})$, a Markov kernel K is called a regular conditional distribution of Y given $\mathbf{X}$, if (and only if) for every set $E \in \mathcal{B}(\mathbb{R})$ the identity

$$K(\mathbf{X}(\omega), E) = \mathbb{E}(\mathbf{1}_E \circ Y | \mathbf{X})(\omega)$$

holds for $\mathbb{P}$-almost every $\omega \in \Omega$. It is a well-known fact, see Kallenberg (2021), that for each pair $(\mathbf{X}, Y)$ such a regular conditional distribution K of Y given $\mathbf{X}$ exists and that it is unique for $\mathbb{P}^{\mathbf{X}}$-almost every $\mathbf{x} \in \mathbb{R}^{d-1}$. For $(\mathbf{X}, Y) \sim A$ we will let $K_A : [0,1]^{d-1} \times \mathcal{B}([0,1]) \to [0,1]$ denote (a version of) the corresponding conditional distribution of Y given $\mathbf{X}$ (i.e., we view K_A directly as a mapping from $[0,1]^{d-1} \times \mathcal{B}([0,1])$ to $[0,1]$); K_A will simply be referred to as (a version of) the $(d-1)$-*Markov kernel* of the copula A.

For every $G \subseteq [0,1]^d$ and $\mathbf{x} \in [0,1]^{d-1}$ define the $\mathbf{x}$-section $G_{\mathbf{x}}$ of G by $G_{\mathbf{x}} := \{y \in [0,1] : (\mathbf{x}, y) \in G\} \in \mathcal{B}([0,1])$. Applying *disintegration* of μ_A into the marginal $\mu_{A^{1:d-1}}$ and the $(d-1)$-Markov kernel K_A of A, see Kallenberg (2021), Section 5, the following identity holds for all $G \in \mathcal{B}([0,1]^d)$:

$$\mu_A(G) = \int_{\mathbb{I}^{d-1}} K_A(\mathbf{x}, G_{\mathbf{x}}) \, \mathrm{d}\mu_{A^{1:d-1}}(\mathbf{x}). \tag{1}$$

We say that a mapping $g : [0,1]^{d-1} \to [0,1]$ is $\mu_{A^{1:d-1}}$-λ preserving, if g is Borel measurable and fulfills that the push-forward $(\mu_{A^{1:d-1}})^g$ of $\mu_{A^{1:d-1}}$ via g coincides with λ. In accordance with Griessenberger et al. (2022) we will call a copula $A \in \mathcal{C}_d$ *completely dependent* (on the first $(d-1)$ coordinates), if there exists some $\mu_{A^{1:d-1}}$-λ-preserving transformation g such that $K(\mathbf{x}, F) := \mathbf{1}_F(g(\mathbf{x}))$ is a regular conditional distribution of A. For $(\mathbf{X}, Y) \sim A$ we have that complete dependence of A is equivalent to the existence of some $\mu_{A^{1:d-1}}$-λ preserving transformation g such that $Y = g(\mathbf{X})$ almost surely.

In the sequel we will work with the family $\mathcal{C}_d^{\Pi_{d-1}}$ of all d-dimensional copulas whose $(d-1)$-dimensional marginals are all equal to Π_{d-1}. Letting d_∞ denote the uniform distance on $\mathcal{C}_d^{\Pi_{d-1}}$ it is straightforward to verify that $(\mathcal{C}_d^{\Pi_{d-1}}, d_\infty)$ is a convex compact set. Following Griessenberger et al. (2022), considering the so-called linkage family

$$\hat{\mathcal{C}}_d := \{A \in \mathcal{C}_d : A^{1:d-1} = \Pi_{d-1}\} \supseteq \mathcal{C}_d^{\Pi_{d-1}}$$

and setting

$$D_1(A, B) = \int_{[0,1]} \int_{[0,1]^{d-1}} \big|K_A(\mathbf{x}, [0, y]) - K_B(\mathbf{x}, [0, y])\big| d\lambda_{d-1}(\mathbf{x}) d\lambda(y) \quad (2)$$

defines a metric on $\hat{\mathcal{C}}_d$. The resulting metric space $(\hat{\mathcal{C}}_d, D_1)$ is complete and separable, convergence w.r.t. D_1 implies convergence w.r.t. to d_∞ but not vice versa. The metric D_1 was introduced for quantifying the extent of directed dependence (again see Griessenberger et al. (2022) and the underlying bivariate construction in Trutschnig (2011)) and therefore (contrary to d_∞) exhibits the property that $D_1(A, \Pi_d)$ is maximal if and only if A is completely dependent, and the maximum is given by $\frac{1}{3}$. Furthermore, the diameter of the metric space $(\hat{\mathcal{C}}_d, D_1)$ is $\frac{1}{2}$.

If $B = \bigtimes_{j \in [d]} [x_j, y_j] \subseteq [0,1]^d$ is a d-dimensional rectangle, its *vertices* will be denoted by

$$\mathrm{ver}(B) = \bigtimes_{j \in [d]} \{x_j, y_j\}.$$

For every vertex $\mathbf{v} \in \mathrm{ver}(B)$ let $m(\mathbf{v}) = |\{j \in [d] \colon v_j = x_j\}|$ and define the *sign* of vertex $\mathbf{v}$ of rectangle B by $\mathrm{sgn}_B(\mathbf{v}) = (-1)^{m(\mathbf{v})}$. For every copula $A \in \mathcal{C}_d$ we have

$$\mu_A(B) = \sum_{\mathbf{v} \in \mathrm{ver}(B)} \mathrm{sgn}_B(\mathbf{v}) A(\mathbf{v}) \geqslant 0. \quad (3)$$

If A is only a d-dimensional quasi-copula, then we will define $\mu_A(B)$ according to eq. (3) – in this case $\mu_A(B)$ may also be negative and μ_A does not necessarily extend to a (signed) measure, see Durante et al. (2016).

3. Main results

Our main objective is to calculate the quantities

$$\Delta_{d_\infty} := \sup\Big\{d_\infty(A,\Pi_d) :\ A \in \mathcal{C}_d^{\Pi_{d-1}}\Big\}, \tag{4}$$

$$\Delta_{D_1} := \sup\Big\{D_1(A,\Pi_d) :\ A \in \mathcal{C}_d^{\Pi_{d-1}}\Big\}, \tag{5}$$

and to show that both suprema are attainable. First we focus on the second one, and start with a simple lemma which is valid in the much more general setting of Haar measure on compact groups, see Rudin (1990), and the proof of which we include for the sake of completeness.

Lemma 3.1. *Suppose that X and Y are independent random variables. If X is uniformly distributed on $[0,1]$, then so is the random variable Z given by*

$$Z := X + Y \bmod (1).$$

Proof. For every $a \in [0,1)$ define the shift $r_a : [0,1] \to [0,1]$ by $r_a(x) = (x+a) \bmod (1)$. Then obviously r_a preserves the Lebesgue measure λ, i.e., the push-forward λ^{r_a} of λ via r_a coincides with λ. Let $E \in \mathcal{B}([0,1])$ be arbitrary but fixed. Then using independence of X and Y, Fubini's theorem, change of coordinates, and $\lambda^{r_y} = \lambda$ (in this order) yields

$$\begin{aligned}
\mathbb{P}(Z \in E) &= \int_\Omega \mathbf{1}_E(Z)d\mathbb{P} = \int_{[0,1]\times\mathbb{R}} \mathbf{1}_E(r_y(x))d\mathbb{P}^{X,Y}(x,y)\\
&= \int_{[0,1]}\int_{\mathbb{R}} \mathbf{1}_E(r_y(x))d\mathbb{P}^{Y}(y)d\mathbb{P}^{X}(x) = \int_{\mathbb{R}}\int_{[0,1]} \mathbf{1}_E(r_y(x))d\mathbb{P}^{X}(x)d\mathbb{P}^{Y}(y)\\
&= \int_{\mathbb{R}}\int_{[0,1]} \mathbf{1}_E(r_y(x))d\lambda(x)d\mathbb{P}^{Y}(y) = \int_{\mathbb{R}}\int_{[0,1]} \mathbf{1}_E(v)d\lambda^{r_y}(v)d\mathbb{P}^{Y}(y)\\
&= \int_{\mathbb{R}} \lambda^{r_y}(E)d\mathbb{P}^{Y}(y) = \int_{\mathbb{R}} \lambda(E)d\mathbb{P}^{Y}(y) = \lambda(E).
\end{aligned}$$

This shows that $\mathbb{P}^Z$ coincides with λ on $[0,1]$ and the proof is complete. □

Taking into account that the diameter of $(\hat{\mathcal{C}}_d, D_1)$ is $\frac{1}{2}$ the following theorem shows that $\mathcal{C}_d^{\Pi_{d-1}} \subseteq \hat{\mathcal{C}}_d$ contains elements that are surprisingly 'far away' from Π_d, in fact, as far away as any element of $\hat{\mathcal{C}}_d$ can be (with respect to D_1).

Theorem 3.2. *For every $d \geqslant 3$ we have $\Delta_{D_1} = \frac{1}{3}$ and the supremum is attained.*

Proof. Considering the afore-mentioned properties of D_1, it suffices to show that the class $\mathcal{C}_d^{\Pi_{d-1}}$ contains completely dependent copulas, which can be done as follows. Let $X_1, \ldots, X_{d-1}$ denote independent, uniformly $[0,1]$-distributed random variables and set

$$X_d := (X_1 + X_2 + \ldots + X_{d-1}) \bmod (1). \tag{6}$$

Then, as a direct consequence of Lemma 3.1, X_d is uniformly distributed on $[0,1]$ too. Letting A denote the copula of the vector $\mathbf{X} = (X_1, \ldots, X_d)$ we obviously have that $A \in \hat{\mathcal{C}}_d$ and that A is completely dependent. In fact, the vector $\mathbf{X}$ even fulfills that each coordinate X_j is a function of the other $(d-1)$ coordinates, i.e., a function of $\mathbf{X}_{-j}$. In other words, A is completely dependent in each direction. To complete the proof it suffices to show that all $(d-1)$-dimensional marginals of A coincide with Π_{d-1}, which can be done as follows. Considering disintegration (w.r.t. the coordinates $2, \ldots, d-1$) and again using $\lambda^{r_a} = \lambda$, for arbitrary $E_2, E_3, \ldots, E_d \in \mathcal{B}([0,1])$ we get

$$\begin{aligned}
\mathbb{P}^{\mathbf{X}_{-1}}\left(\bigtimes_{i=2}^{d} E_i\right) &= \mathbb{P}^{\mathbf{X}}\left([0,1] \times \bigtimes_{i=2}^{d} E_i\right) \\
&= \int_{\times_{i=2}^{d-1} E_i} \mathbb{P}\left(X_1 \in [0,1], r_{\sum_{i=2}^{d-1} x_i \bmod (1)}(X_1) \in E_d\right) d\lambda_{d-2}(\mathbf{x}_{2:d-1}) \\
&= \int_{\times_{i=2}^{d-1} E_i} \mathbb{P}\left(r_{\sum_{i=2}^{d-1} x_i \bmod (1)}(X_1) \in E_d\right) d\lambda_{d-2}(\mathbf{x}_{2:d-1}) \\
&= \int_{\times_{i=2}^{d-1} E_i} \lambda(E_d)\, d\lambda_{d-2}(\mathbf{x}_{2:d-1}) = \prod_{i=2}^{d} \lambda(E_i) = \prod_{i=2}^{d} \mathbb{P}^{X_i}(E_i).
\end{aligned}$$

This shows $\mathbf{X}_{-1} \sim \Pi_{d-1}$. Replacing $j = 1$ by any other index $j \in \{2, \ldots, d-1\}$ and proceeding analogously yields $\mathbf{X}_{-j} \sim \Pi_{d-1}$. Since $\mathbf{X}_{-d} \sim \Pi_{d-1}$ by construction, the proof is complete. The left panel of Figure 1 depicts a sample of the copula A for the case $d = 3$. □

Our proof of Theorem 3.2 establishes a slightly stronger result – before stating it, we fix some notation: Σ_d denotes the set of all $d!$ permutations of $[d] = \{1, \ldots, d\}$; for every permutation $\varepsilon \in \Sigma_d$ and $A \in \mathcal{C}_d^{\Pi_{d-1}}$ we let A^ε denote the copula obtained from A by permuting the coordinates according to ε, i.e., A^ε is the copula of the vector $\mathbf{X}_{\varepsilon(\mathbf{i})}$ where $\mathbf{X} \sim A$.

Corollary 3.3. *For every $d \geqslant 3$ we have*

$$\sup\left\{ \min\{D_1(A^{\varepsilon}, \Pi_d) : \varepsilon \in \Sigma_d\} : A \in \mathcal{C}_d^{\Pi_{d-1}} \right\} = \tfrac{1}{3}$$

and the supremum is attained.

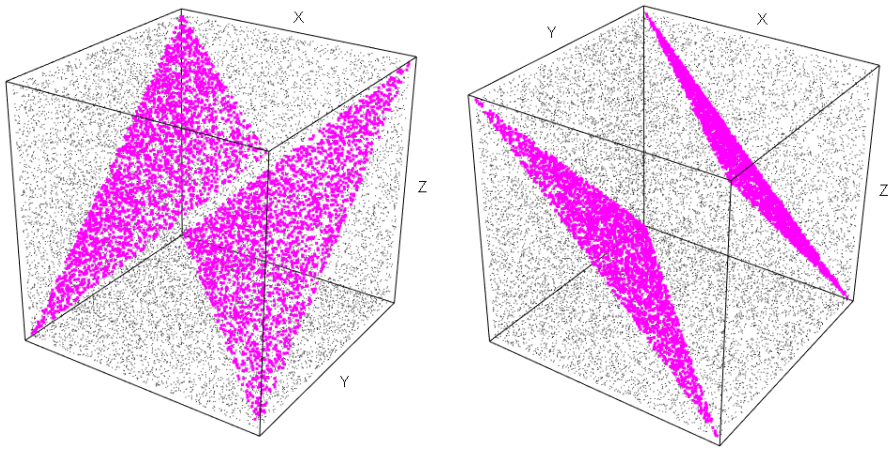

Figure 1: Left panel: Sample of size $n = 5.000$ of the copula $A \in \mathcal{C}_3^{\Pi_2}$ (in dimension $d = 3$) as considered in the proof of Theorem 3.2 in magenta, bivariate marginal samples in gray (i.e., the three-dimensional magenta points are projected onto the hyperplanes $h_1 : z = 0$, $h_2 : y = 1$, $h_3 : x = 0$).
Right panel: Sample of the symmetric copula $B \in \mathcal{C}_3^{\Pi_2}$ considered in the second part of Remark 3.4.

Remark 3.4. Notice that fixing $a \in (0, 1)$, slightly changing the definition of X_d according to eq. (6) to

$$X_d := (X_1 + X_2 + \ldots + X_{d-1} + a) \bmod (1),$$

and proceeding like in the proof of Theorem 3.2 results in another copula $A_a \in \mathcal{C}_d^{\Pi_{d-1}}$ which is completely dependent and which also fulfills $D_1(A_a, \Pi_d) = \frac{1}{3} = \Delta_{D_1}$. In other words, we can easily construct uncountably many elements in $\mathcal{C}_d^{\Pi_{d-1}}$ having D_1-distance $\frac{1}{3}$ to Π_d. Moreover, our construction can easily be modified in order to obtain symmetric (exchangeable) copulas with the afore-mentioned maximization property w.r.t. D_1. In fact, replacing (6) by

$$X_d := 1 - \big((X_1 + X_2 + \ldots + X_{d-1}) \bmod (1)\big),$$

we obviously have $(\sum_{i=1}^{d} X_i) \bmod (1) = 0$ almost surely and for every $j \in [d]$ the identity

$$X_j = 1 - \left(\left(\sum_{i \neq j} X_i \right) \bmod (1) \right)$$

holds almost surely. As a direct consequence, the copula B of $\mathbf{X} = (X_1, \ldots, X_d)$ is exchangeable and fulfills $D_1(B, \Pi_d) = \frac{1}{3} = \Delta_{D_1}$. The right panel of Figure 1 depicts a sample (plus bivariate marginal samples) of the copula B for $d = 3$.

We now turn our attention to Δ_{d_∞} and first derive a pointwise lower and upper bound for any copula $A \in \mathcal{C}_d^{\Pi_{d-1}}$. Such bounds have already been considered in Deheuvels (1983), but the bounds derived in Proposition 3.5 are stronger (they are, in fact, best possible). Furthermore, Joe (1997) considered bounds for d-variate distribution functions with given $(d-1)$-variate marginals in the case $d = 3, 4$. In case the marginals are uniform, his formulas simplify to the bounds in Proposition 3.5.

Proposition 3.5. *Let $d \geqslant 2$, $A \in \mathcal{C}_d^{\Pi_{d-1}}$ and $\mathbf{u} \in [0,1]^d$. Then*

$$S_d(\mathbf{u}) \leqslant A(\mathbf{u}) \leqslant T_d(\mathbf{u}),$$

where

$$S_d(\mathbf{u}) = \Pi_d(\mathbf{u}) - \min_{\substack{J \subseteq [d] \\ |J| \text{ even}}} \left\{ \prod_{j \in J} (1 - u_j) \prod_{j \in [d] \setminus J} u_j \right\}, \text{ and}$$

$$T_d(\mathbf{u}) = \Pi_d(\mathbf{u}) + \min_{\substack{J \subseteq [d] \\ |J| \text{ odd}}} \left\{ \prod_{j \in J} (1 - u_j) \prod_{j \in [d] \setminus J} u_j \right\}.$$

Moreover, for every $\mathbf{u} \in [0,1]^d$ there exist copulas $A_1, A_2 \in \mathcal{C}_d^{\Pi_{d-1}}$ such that $A_1(\mathbf{u}) = S_d(\mathbf{u})$ and $A_2(\mathbf{u}) = T_d(\mathbf{u})$.

Proof. For every subset $J \subseteq [d]$ and every index $j \in [d]$ we denote by E_j^J the interval $[u_j, 1]$ if $j \in J$ and the interval $[0, u_j]$ if $j \in [d] \setminus J$. Furthermore, let B_J be the rectangle $B_J = \bigtimes_{j \in [d]} E_j^J$ and let

$$\operatorname{ver}(B_J) \subseteq \Delta := \bigtimes_{j \in [d]} \{0, u_j, 1\}$$

be the set of vertices of B_J. Note that $\mathbf{u} \in \operatorname{ver}(B_J)$ and $\operatorname{sgn}_{B_J}(\mathbf{u}) = (-1)^{|J|}$. Since $A \in \mathcal{C}_d^{\Pi_{d-1}}$, we have $A(\mathbf{v}) = \Pi_d(\mathbf{v})$ for every $\mathbf{v} \in \operatorname{ver}(B_J)$, except for $\mathbf{v} = \mathbf{u}$. This yields

$$\begin{aligned} 0 \leqslant \mu_A(B_J) &= \sum_{\mathbf{v} \in \operatorname{ver}(B_J)} \operatorname{sgn}_{B_J}(\mathbf{v}) A(\mathbf{v}) \\ &= \sum_{\mathbf{v} \in \operatorname{ver}(B_J)} \operatorname{sgn}_{B_J}(\mathbf{v}) \Pi_d(\mathbf{v}) - \operatorname{sgn}_{B_J}(\mathbf{u}) \Pi_d(\mathbf{u}) + \operatorname{sgn}_{B_J}(\mathbf{u}) A(\mathbf{u}) \\ &= \mu_{\Pi_d}(B_J) - (-1)^{|J|} \Pi_d(\mathbf{u}) + (-1)^{|J|} A(\mathbf{u}) \\ &= \prod_{j \in J} (1 - u_j) \prod_{j \in [d] \setminus J} u_j - (-1)^{|J|} \Pi_d(\mathbf{u}) + (-1)^{|J|} A(\mathbf{u}). \end{aligned}$$

If $|J|$ is odd, it follows that

$$A(\mathbf{u}) \leqslant \Pi_d(\mathbf{u}) + \prod_{j \in J} (1 - u_j) \prod_{j \in [d] \setminus J} u_j,$$

if $|J|$ is even, we obtain

$$A(\mathbf{u}) \geqslant \Pi_d(\mathbf{u}) - \prod_{j \in J} (1 - u_j) \prod_{j \in [d] \setminus J} u_j.$$

Collecting all obtained inequalities, we infer

$$A(\mathbf{u}) \leqslant \min_{\substack{J \subseteq [d] \\ |J| \text{ odd}}} \left\{ \Pi_d(\mathbf{u}) + \prod_{j \in J} (1 - u_j) \prod_{j \in [d] \setminus J} u_j \right\} = T_d(\mathbf{u}),$$

as well as

$$A(\mathbf{u}) \geqslant \max_{\substack{J \subseteq [d] \\ |J| \text{ even}}} \left\{ \Pi_d(\mathbf{u}) - \prod_{j \in J} (1 - u_j) \prod_{j \in [d] \setminus J} u_j \right\} = S_d(\mathbf{u}).$$

It remains to show that the bounds are attained. Let $\mathbf{u} \in [0,1]^d$ be arbitrary but fixed and define a function $\widetilde{A}_1 \colon \Delta \to [0,1]$ by

$$\widetilde{A}_1(\mathbf{v}) = \begin{cases} S_d(\mathbf{u}); & \text{if } \mathbf{v} = \mathbf{u}, \\ \Pi_d(\mathbf{v}); & \text{otherwise.} \end{cases}$$

For every subset $J \subseteq [d]$ we have

$$\begin{aligned} \sum_{\mathbf{v} \in \operatorname{ver}(B_J)} \operatorname{sgn}_{B_J}(\mathbf{v}) \widetilde{A}_1(\mathbf{v}) &= \sum_{\mathbf{v} \in \operatorname{ver}(B_J)} \operatorname{sgn}_{B_J}(\mathbf{v}) \Pi_d(\mathbf{v}) - \operatorname{sgn}_{B_J}(\mathbf{u}) (\Pi_d(\mathbf{u}) - S_d(\mathbf{u})) \\ &= \prod_{j \in J} (1 - u_j) \prod_{j \in [d] \setminus J} u_j - (-1)^{|J|} (\Pi_d(\mathbf{u}) - S_d(\mathbf{u})). \end{aligned}$$

If $|J|$ is even, it follows that

$$\sum_{\mathbf{v}\in\mathrm{ver}(B_J)} \mathrm{sgn}_{B_J}(\mathbf{v})\widetilde{A}_1(\mathbf{v}) = \prod_{j\in J}(1-u_j) \prod_{j\in[d]\setminus J} u_j - \Pi_d(\mathbf{u}) + S_d(\mathbf{u}) \geqslant 0$$

by the definition of S_d. If $|J|$ is odd, we have

$$\begin{aligned}\sum_{\mathbf{v}\in\mathrm{ver}(B_J)} \mathrm{sgn}_{B_J}(\mathbf{v})\widetilde{A}_1(\mathbf{v}) &= \prod_{j\in J}(1-u_j) \prod_{j\in[d]\setminus J} u_j + \Pi_d(\mathbf{u}) - S_d(\mathbf{u}) \\ &\geqslant \prod_{j\in J}(1-u_j) \prod_{j\in[d]\setminus J} u_j \geqslant 0,\end{aligned}$$

since $S_d(\mathbf{u}) \leqslant \Pi_d(\mathbf{u})$. It follows that $\widetilde{A}_1$ is a subcopula which we can extend to a copula A_1, defined on entire $[0,1]^d$, via (piecewise) multilinear extension on each rectangle B_J (see Nelsen (2006) for the definition of subcopulas and their extensions). Since $\widetilde{A}_1(\mathbf{v}) = \Pi_d(\mathbf{v})$ for every $\mathbf{v} \in \Delta$ lying on the boundary of $[0,1]^d$, the $(d-1)$-dimensional marginals of A_1 are equal to Π_{d-1}, so $A_1 \in \mathcal{C}_d^{\Pi_{d-1}}$. Furthermore, $A_1(\mathbf{u}) = S_d(\mathbf{u})$ by construction. Finally, defining a function $\widetilde{A}_2 \colon \Delta \to [0,1]$ by

$$\widetilde{A}_2(\mathbf{v}) = \begin{cases} T_d(\mathbf{u}); & \text{if } \mathbf{v} = \mathbf{u}, \\ \Pi_d(\mathbf{v}); & \text{otherwise,} \end{cases}$$

showing that it is a subcopula similarly as above, and extending it to a copula $A_2 \in \mathcal{C}_d^{\Pi_{d-1}}$, satisfying $A_2(\mathbf{u}) = T_d(\mathbf{u})$, completes the proof. □

Note that in the case $d = 2$ the bounds S_2 and T_2 are equal to the Fréchet-Hoeffding bounds W_2 and M_2. In the case $d = 3$ the bounds in Proposition 3.5 coincide with functions S_3 and T_3 given in Nelsen and Úbeda-Flores (2012). Furthermore, being pointwise infima and suprema of copulas, the functions S_d and T_d are quasi-copulas and, for $d \geqslant 3$, not copulas, as the following lemma shows. For $d = 3$ this was already proved in Nelsen and Úbeda-Flores (2012).

Lemma 3.6. *Let $d \geqslant 3$ and consider the rectangle $B = [\frac{1}{3}, \frac{2}{3}]^3 \times [0, \frac{1}{3}]^{d-3}$ in $[0,1]^d$. Then*

$$\mu_{S_d}(B) = \mu_{T_d}(B) = -\tfrac{1}{3^{d-1}} < 0$$

holds, so both S_d and T_d are proper quasi-copulas.

Proof. Consider

$$\widetilde{S}_d(\mathbf{u}) := \Pi_d(\mathbf{u}) - S_d(\mathbf{u}) = \min_{\substack{J\subseteq[d]\\ |J| \text{ even}}} \Big\{ \prod_{j\in J}(1-u_j) \prod_{j\in[d]\setminus J} u_j \Big\},$$
$$\widetilde{T}_d(\mathbf{u}) := T_d(\mathbf{u}) - \Pi_d(\mathbf{u}) = \min_{\substack{J\subseteq[d]\\ |J| \text{ odd}}} \Big\{ \prod_{j\in J}(1-u_j) \prod_{j\in[d]\setminus J} u_j \Big\}.$$

and set

$$\begin{aligned}
\mathbf{u}_3 &:= (\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}, \dots, \tfrac{1}{3}),\\
\mathbf{u}_2 &:= (\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \dots, \tfrac{1}{3}),\\
\mathbf{u}_1 &:= (\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \dots, \tfrac{1}{3}), \text{ and}\\
\mathbf{u}_0 &:= (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \dots, \tfrac{1}{3})
\end{aligned}$$

Then all four points $\mathbf{u}_0, \dots, \mathbf{u}_3$ are vertices of the rectangle B and all other vertices of B are either obtained by permuting the first three coordinates of these vertices or contain at least one coordinate equal to 0. We claim that

$$\widetilde{S}_d(\mathbf{u}_3) = \tfrac{2}{3^d}, \quad \widetilde{S}_d(\mathbf{u}_2) = \tfrac{1}{3^d}, \quad \widetilde{S}_d(\mathbf{u}_1) = \tfrac{2}{3^d}, \quad \text{and} \quad \widetilde{S}_d(\mathbf{u}_0) = \tfrac{1}{3^d}.$$

Indeed, this is straightforward for $d = 3$. If $d > 3$, the minimum over sets $J \subseteq [d]$ of even cardinality is always achieved by a set J with $J \cap \{4, ..., d\} = \varnothing$, since including any coordinate from $\{4, ..., d\}$ in J replaces a factor of $\frac{1}{3}$ by $\frac{2}{3}$, so strictly increases the product. So, the value of $\widetilde{S}_d$ at $\mathbf{u}_k$ is obtained by multiplying the value of $\widetilde{S}_3$ at corresponding $\mathbf{u}_k$ by $\frac{1}{3^{d-3}}$. It follows that

$$\mu_{\widetilde{S}_d}(B) = \widetilde{S}_d(\mathbf{u}_3) - 3\widetilde{S}_d(\mathbf{u}_2) + 3\widetilde{S}_d(\mathbf{u}_1) - \widetilde{S}_d(\mathbf{u}_0) = \tfrac{4}{3^d},$$

since the remaining vertices of B contain a zero coordinate, hence do not contribute to the signed volume. This implies

$$\mu_{S_d}(B) = \mu_{\Pi_d}(B) - \mu_{\widetilde{S}_d}(B) = \tfrac{1}{3^d} - \tfrac{4}{3^d} = -\tfrac{1}{3^{d-1}}.$$

Using similar arguments as above, we obtain

$$\widetilde{T}_d(\mathbf{u}_3) = \tfrac{1}{3^d}, \quad \widetilde{T}_d(\mathbf{u}_2) = \tfrac{2}{3^d}, \quad \widetilde{T}_d(\mathbf{u}_1) = \tfrac{1}{3^d}, \quad \text{and} \quad \widetilde{T}_d(\mathbf{u}_0) = \tfrac{2}{3^d},$$

so $\mu_{\widetilde{T}_d}(B) = -\frac{4}{3^d}$, and $\mu_{T_d}(B) = \mu_{\Pi_d}(B) + \mu_{\widetilde{T}_d}(B) = -\frac{1}{3^{d-1}}$. □

Now, we compute the value of Δ_{d_∞} for every d.

Theorem 3.7. *For every $d \geqslant 2$ we have $\Delta_{d_\infty} = \frac{1}{2^d}$ and the supremum is attained.*

Proof. For every $\mathbf{u} \in [0,1]^d$, using inequality $\min\{a_1, a_2, \ldots, a_m\} \leqslant \sqrt[m]{\prod_{i=1}^m a_i}$, we have

$$T_d(\mathbf{u}) - \Pi_d(\mathbf{u}) = \min_{\substack{J\subseteq[d]\\ |J| \text{ odd}}} \Big\{ \prod_{j\in J}(1-u_j) \prod_{j\in[d]\setminus J} u_j \Big\} \leqslant \sqrt[2^{d-1}]{\prod_{\substack{J\subseteq[d]\\ |J| \text{ odd}}} \Big(\prod_{j\in J}(1-u_j) \prod_{j\in[d]\setminus J} u_j \Big)}.$$

Note that every $j \in [d]$ appears in precisely 2^{d-2} subsets $J \subseteq [d]$ of odd cardinality (and does not appear in the remaining 2^{d-2} subsets $J \subseteq [d]$ of odd cardinality). This implies

$$T_d(\mathbf{u}) - \Pi_d(\mathbf{u}) \leqslant \sqrt[2^{d-1}]{\prod_{j\in[d]} (1-u_j)^{2^{d-2}} \prod_{j\in[d]} u_j^{2^{d-2}}} = \sqrt{\prod_{j\in[d]} (1-u_j)u_j} \leqslant \tfrac{1}{2^d},$$

since $(1-u_j)u_j \leqslant \frac{1}{4}$ for every j. Similarly, for every $\mathbf{u} \in [0,1]^d$

$$\Pi_d(\mathbf{u}) - S_d(\mathbf{u}) = \min_{\substack{J\subseteq[d]\\ |J| \text{ even}}} \Big\{ \prod_{j\in J}(1-u_j) \prod_{j\in[d]\setminus J} u_j \Big\} \leqslant \tfrac{1}{2^d}.$$

It hence follows that

$$\begin{aligned}
\Delta_{d_\infty} &= \sup\Big\{ d_\infty(A, \Pi_d) : \ A \in \mathcal{C}_d^{\Pi_{d-1}} \Big\} \\
&= \sup\Big\{ \sup\{|A(\mathbf{u}) - \Pi_d(\mathbf{u})| : \ \mathbf{u} \in [0,1]^d\} : \ A \in \mathcal{C}_d^{\Pi_{d-1}} \Big\} \\
&= \sup\Big\{ \sup\big\{|A(\mathbf{u}) - \Pi_d(\mathbf{u})| : \ A \in \mathcal{C}_d^{\Pi_{d-1}}\big\} : \ \mathbf{u} \in [0,1]^d \Big\} \\
&= \sup\Big\{ \max\{T_d(\mathbf{u}) - \Pi_d(\mathbf{u}), \Pi_d(\mathbf{u}) - S_d(\mathbf{u})\} : \ \mathbf{u} \in [0,1]^d \Big\} \\
&\leqslant \tfrac{1}{2^d}.
\end{aligned}$$

On the other hand, let $\mathbf{u}_0 = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \in [0,1]^d$, to estimate

$$\begin{aligned}
\Delta_{d_\infty} &= \sup\Big\{ \max\{T_d(\mathbf{u}) - \Pi_d(\mathbf{u}), \Pi_d(\mathbf{u}) - S_d(\mathbf{u})\} : \ \mathbf{u} \in [0,1]^d \Big\} \\
&\geqslant T_d(\mathbf{u}_0) - \Pi_d(\mathbf{u}_0) = \min_{\substack{J\subseteq[d]\\ |J| \text{ odd}}} \Big\{ \prod_{j\in J}(1-\tfrac{1}{2}) \prod_{j\in[d]\setminus J} \tfrac{1}{2} \Big\} \\
&= \min_{\substack{J\subseteq[d]\\ |J| \text{ odd}}} \{\tfrac{1}{2^d}\} = \tfrac{1}{2^d}.
\end{aligned}$$

This shows that the supremum is attained and finishes the proof. □

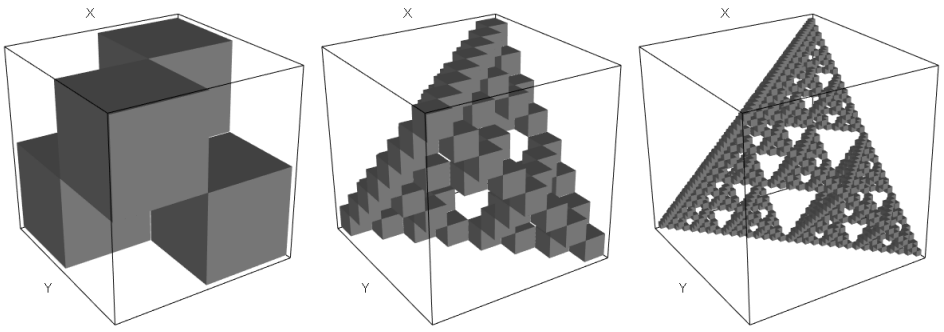

Figure 2: Steps 1, 3 and 5 in the IFSP-based construction of the completely dependent copula $A \in \mathcal{C}_3^{\Pi_2}$ fulfilling $d_\infty(A, \Pi_3) = \frac{1}{2^3} = \Delta_{d_\infty}$ as well as $D_1(A, \Pi_3) = \frac{1}{3} = \Delta_{D_1}$.

Example 3.8. Although the metrics d_∞ and D_1 are conceptually quite different, there are copulas $A \in \mathcal{C}_d^{\Pi_{d-1}}$ fulfilling that they maximize both Δ_{d_∞} and Δ_{D_1}. In fact, considering $d = 3$ and working with Iterated Function Systems with Probabilities (IFSP) induced by so-called generalized transformation matrices, see Dietrich et al. (2026), it is possible to construct a copula $A \in \mathcal{C}_3^{\Pi_2}$ which is completely dependent and which fulfills $A(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = 0$, implying $d_\infty(A, \Pi_3) \geqslant |0 - \frac{1}{2^3}| = \frac{1}{2^3} = \Delta_{d_\infty}$. Figure 2 depicts the stepwise IFSP approximation of support/density of this copula A.

Remark 3.9. The conceptual difference of the metrics d_∞ and D_1 shows itself once again in the fact that $\Delta_{d_\infty} = \frac{1}{2^d} \to 0$ for $d \to \infty$ whereas $\Delta_{D_1} = \frac{1}{3}$ for every $d \geqslant 3$. Intuitively, with growing dimension d, the growing number of $(d-1)$-marginal constraints in $\mathcal{C}_d^{\Pi_{d-1}}$ allows for less and less flexibility in the sense of pointwise deviation from Π_d (in fact, the flexibility reduces exponentially). By contrast, as mentioned in Section 2, $D_1(A, \Pi_d)$ is maximal if and only if A is completely dependent and the maximal value is given by $\frac{1}{3}$ for every $d \geqslant 3$. Since complete dependence is compatible with uniform $(d-1)$-marginals, $\mathcal{C}_d^{\Pi_{d-1}}$ contains completely dependent elements and the maximum possible D_1-distance to Π_d does not decay with growing dimension d.

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