Angewandte (Mathematische) Statistik (405.330)

Elementary toy example hypothesis testing

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Example (Elementary toy example hypothesis testing)

- Suppose that somebody rolls a dice (that you can not see).
- You only know that the dice either has (i) a '1' on four sides and a '0' on the other two sides or (ii) a '1' on two sides and a '0' on the other four sides.
- If we let X denote the result of rolling this dice once, then we either have

$$(i)\ p := \mathbb{P}(X=1) = \frac{4}{6} = \frac{2}{3} \quad \text{and} \quad \mathbb{P}(X=0) = \frac{2}{6} = \frac{1}{3} = 1-p$$

or

$$(ii)\ p := \mathbb{P}(X=1) = \frac{2}{6} = \frac{1}{3} \quad \text{and} \quad \mathbb{P}(X=0) = \frac{4}{6} = \frac{2}{3} = 1-p.$$

- In other words, $X \sim A(p)$ and we know that $p \in \Theta = \{\frac{2}{3}, \frac{1}{3}\}$.
- We will call $H_0 : p = \frac{2}{3}$ the *null hypothesis* and $H_1 : p = \frac{1}{3}$ the *alternative hypothesis* (for whatever reason).

Example (Toy example hypothesis testing, cont.)

- For the moment we focus on $H_0 : p = \frac{2}{3}$.
- Suppose that the dice is rolled twice and the result is denoted by (X_1, X_2).
- Possibility 1: $(X_1, X_2) = (1, 1)$. Would you stick to H_0 or reject H_0 (i.e. change to H_1), and why?
- Possibility 2: $(X_1, X_2) = (1, 0)$. Would you stick to H_0 or reject H_0, and why?
- Possibility 3: $(X_1, X_2) = (0, 1)$. Would you stick to H_0 or reject H_0, and why?
- Possibility 4: $(X_1, X_2) = (0, 0)$. Would you stick to H_0 or reject H_0, and why?
- Which criterion is your decision based upon?
- For a given observation we check under which of the two hypotheses the observation has higher probability.

Example (Toy example hypothesis testing, cont.)

- If H_0 is correct then we have

$$\begin{aligned}\mathbb{P}_{H_0}(X_1 = 1, X_2 = 1) &= \frac{4}{9}, \quad \mathbb{P}_{H_0}(X_1 = 1, X_2 = 0) = \frac{2}{9} \\ \mathbb{P}_{H_0}(X_1 = 0, X_2 = 1) &= \frac{2}{9}, \quad \mathbb{P}_{H_0}(X_1 = 0, X_2 = 0) = \frac{1}{9}.\end{aligned}$$

- If H_1 is correct then we have

$$\begin{aligned}\mathbb{P}_{H_1}(X_1 = 1, X_2 = 1) &= \frac{1}{9}, \quad \mathbb{P}_{H_1}(X_1 = 1, X_2 = 0) = \frac{2}{9} \\ \mathbb{P}_{H_1}(X_1 = 0, X_2 = 1) &= \frac{2}{9}, \quad \mathbb{P}_{H_1}(X_1 = 0, X_2 = 0) = \frac{4}{9}.\end{aligned}$$

- In case of $(1,1)$ we do not reject H_0.
- In case of $(1,0)$ and in case of $(0,1)$ we do not reject H_0 (the observation is equally probable under both hypotheses, so by changing from H_0 to H_1 we don't gain anything).
- In case of $(0,0)$ we reject H_0.

Example (Toy example hypothesis testing, cont.)

- We intuitively reject H_0 if - under the assumption that H_0 is true - the observation we made is very unlikely (in the sense of having low probability).
- In our toy setting we can make two different mistakes:
- **Type I error**: We reject H_0 although it is correct.
- **Type II error**: We do not reject (accept) H_0 although it is wrong.
- Let us calculate the probability of a type I and the probability of a type II error in our toy setting:
- @type I error α:

$$\alpha := \mathbb{P}_{H_0}(\text{reject } H_0) = \mathbb{P}_{H_0}(X_1 = 0, X_2 = 0) = \frac{1}{9}$$

- We have a chance of more than 11% to make a type I error.

Example (Toy example hypothesis testing, cont.)

- @type II error β:

$$\begin{aligned}\beta := \mathbb{P}_{H_1}(\text{accept } H_0) &= \mathbb{P}_{H_1}(X_1 = 1, X_2 = 1) + \mathbb{P}_{H_1}(X_1 = 1, X_2 = 0) \\ &\quad + \mathbb{P}_{H_1}(X_1 = 0, X_2 = 1) \\ &= 1 - \mathbb{P}_{H_1}(X_1 = 0, X_2 = 0) = \frac{5}{9}\end{aligned}$$

- We have chance of more than 55% to make a type II error.
- Could we improve our decision criterion to reduce the type I and the type II error?
- Is there a perfect decision rule such that $\alpha = \beta = 0$?
- If we want $\alpha = 0$ then we can NEVER reject H_0, so we get $\beta = 1$.
- If we want $\beta = 0$ then we always have to reject H_0, so we get $\alpha = 1$.
- α and β are antagonists.
- Which one is more important? Think of a criminal trial...

Hypothesis testing vs. criminal trials

- Consider a criminal trial.
- Based on evidence the jury (or the judge) has to decide whether the defendant is guilty or not.
- Suppose that $H_0 = \{\text{innocent}\}$ and that $H_1 = \{\text{guilty}\}$.
- Right at the start the jury (or the judge) accepts H_0 and assumes that the defendant is innocent.
- Only if enough evidence is brought in, H_0 will be rejected and the defendant will be declared guilty.
- The afore-mentioned type I error α corresponds to the situation that the defendant will be declared guilty although he is innocent.
- The afore-mentioned type II error β corresponds to the situation that the defendant will be declared innocent although he is guilty.

- Which error has worse consequences for the defendant?
- Obviously the type I error.
- In the Anglo-Saxon jurisdiction system there there is the term 'Beyond reasonable doubt' underlining this fact.
- In other words: We want to keep the type I error α (very) small.
- The same applies to hypothesis testing: α should be small, standard *significance levels* are $\alpha = 0.05$ and $\alpha = 0.01$ (one error out of twenty or one out of hundred).
- As soon as α is fixed it is the statisticians' job to develop optimal tests, i.e. decision rules (criteria) with a probability of (at most) α for a type I error and, at the same time, minimal type II error β.

Example (Toy example hypothesis testing, cont.)

- Suppose we fix $\alpha = 0.05$ and want to develop a decision rule (i.e. a criterion when to reject H_0) such that the probability of a type I error is at most 0.05.
- Since, under $H_0 : p = \frac{2}{3}$ all four possible outcomes have at least a probability of $\frac{1}{9}$ the only choice we have is never to reject H_0, in which case $\beta = 1$.
- This looks pretty bad at first sight...keeping in mind, however, the criminal trial comparison it would mean that the jury should not declare the defendant guilty if there is not enough evidence against it (remember: 'Beyond reasonable doubt').
- If, instead of sample size two (two observations), we had sample size $n = 100$ the situation would improve - let's develop a simple test for this situation:
- As before we have $H_0 : p = \frac{2}{3}$ and $H_1 : p = \frac{1}{3}$ and we want the error of type I to be at most 0.05.
- A natural idea is the following: Reject H_0 if the sample $x_1, x_2, \ldots, x_n$ contains 0 too many times or, equivalently, 1 not often enough.

Example (Toy example hypothesis testing, cont.)

- How to determine the threshold t?
- Under H_0 the number K of 1s in the sample of size $n = 100$ has a Binomial distribution $Bin(n, p)$ with parameter $p = \frac{2}{3}$, i.e

$$\mathbb{P}_{H_0}(K = k) = \binom{100}{k}\left(\frac{2}{3}\right)^k\left(\frac{1}{3}\right)^{100-k}.$$

- The threshold t has to fulfill

$$\mathbb{P}_{H_0}(K \leq t) \stackrel{!}{=} 0.05. \tag{1}$$

- There is no exact solution t of equation (1) so we calculate the biggest t fulfilling

$$\mathbb{P}_{H_0}(K \leq t) \leq 0.05 \tag{2}$$

and get $t = 58$ (see R-Code).

Example (Toy example hypothesis testing, cont.)

- Altogether we have arrived at the following test for H_0 vs. H_1 given $n = 100$ observations $x_1, \dots, x_n$:
- Reject H_0 if the number K of 1s in the sample fulfills $K \leq 58$.
- Do not reject H_0 if $K > 58$.
- It follows from the construction (again see R-Code) that

$$\alpha = \mathbb{P}_{H_0}(\text{reject } H_0) = \mathbb{P}_{H_0}(K \leq 58) = 0.04337149,$$

  i.e. in 4.3% of all cases we reject H_0 although it is correct.
- How big is the probability of a type II error?
- We calculate it as before and get

$$\beta = \mathbb{P}_{H_1}(\text{accept } H_0) = \mathbb{P}_{H_1}(K > 58) = 1 - \mathbb{P}_{H_1}(K \leq 58) = 0.00000012907.$$

- How can this be interpreted?

Example (Toy example hypothesis testing, cont.)

- A quick look at the R-Code

```
#determine the threshold for the test H0: p=2/3 versus H1: p=1/3
plot(0:100,pbinom(0:100,size=100,prob=2/3),type="p")
abline(h=0.05)

t<-qbinom(p=0.05,size=100,prob=2/3)-1
t
[1] 58

pbinom(t,size = 100,prob=2/3)
[1] 0.04337149

#calculate beta
1-pbinom(t,size=100,prob=1/3)
[1] 1.290734e-07
```

Example (Toy example hypothesis testing, cont.)

- Let us check if the just developed test really performs as it should - we run simulations (always important especially in the context of hypothesis testing).

```
#evaluate performance of the developed test
# one run under H0:
n<-100
p<-2/3
x<-sample(c(1,0),size=n,replace = TRUE,prob=c(2/3,1/3))
if(length(x[x==1])<=58){print("reject H0")}
```

```

# R=10000 runs under H0
R<-10000
reject<-rep(0,R)
for(i in 1:R){
  x<-sample(c(1,0),size=n,replace = TRUE,prob=c(2/3,1/3))
  if(length(x[x==1])<=58){reject[i]<-1}
}
mean(reject)
[1] 0.0445

barplot(table(reject))
```

Example (Toy example hypothesis testing, cont.)

- Simulations for the type II error.

```
# R=10000 runs under H1
R<-10000
false<-rep(0,R)
for(i in 1:R){
  x<-sample(c(1,0),size=n,replace = TRUE,prob=c(1/3,2/3))
  if(length(x[x==1])>=58){false[i]<-1}
}
mean(false)
[1] 0
```

- The type II error is really (almost) zero, i.e. if $H_1 : p = \frac{1}{3}$ is true, the test detects it (almost) every time.

Exercise:

- Suppose that the toy example is slightly modified as follows:
- You only know that the dice either has (i) a 1 on three sides and a 0 on the other three sides or (ii) a 1 on two sides and a 0 on the other four sides.
- Develop a test with type I error of at most 0.05 for this situation, i.e. a test for $H_0 : p = \frac{1}{2}$ vs. $H_1 : p = \frac{1}{3}$.
- Evaluate the performance of this test by modifying the provided R-Code accordingly.
- Work with different sample sizes, e.g. $n = 10, n = 20, n = 50, n = 100, n = 500$, and describe the influence of the sample size on α and (more importantly) on β.

Quick reminder

- We had an experiment X with a binary output 1 and 0.
- We knew that the success probability $p = \mathbb{P}(X = 1)$ was either $p = \frac{2}{3}$ or $p = \frac{1}{3}$.
- We developed a hypothesis test for $H_0 : p = \frac{2}{3}$ versus $H_1 : p = \frac{1}{3}$ based on samples $x_1, \ldots, x_n$ of size $n = 100$.
- The test we developed at a significance level $\alpha = 0.05$ was to reject H_0 if the number K of ones in $x_1, \ldots, x_n$ fulfills $K \leq 58$.
- The probability of a type I error (what was that?) was $\alpha = \mathbb{P}_{H_0}(K \leq 58) = 0.04337149$.
- The probability of a type II error (what was that?) was $\beta = \mathbb{P}_{H_1}(K > 58) = 0.00000012907$.
- How can these two values be interpreted?

- Assume that H_0 is correct:
- Then out of $R = 10.000$ times we falsely reject H_0 approx. 434 times
- Assume that H_1 is correct:
- Then out of $R = 10.000$ times we do not reject H_0 approx. 0 times
- Remember that α and β can not be minimized simultaneously, so α comes first (criminal trial comparison).

- Suppose we now want to test $H_0 : p \geq \frac{1}{2}$ vs. $H_1 : p < \frac{1}{2}$ at significance level $\alpha = 0.05$.
- Why is this situation more complicated and what is the key difference to $H_0 : p = \frac{2}{3}$ versus $H_1 : p = \frac{1}{3}$?
- H_0 and H_1 are **composite**, i.e. they contain more than one value of the parameter.

- How could we extend the definition of the type I error $\mathbb{P}_{H_0}(\text{reject } H_0)$ to this situation?
- If the true parameter is p then H_0 holds whenever $p \geq \frac{1}{2}$.
- What we want is

$$\mathbb{P}_p(\text{reject } H_0) \leq 0.05 \tag{3}$$

for every $p \geq \frac{1}{2}$.

- Mathematically speaking we want

$$\max_{p \in H_0} \mathbb{P}_p(\text{reject } H_0) \leq 0.05$$

- Does it make sense to proceed analogously with the type II error β and set

$$\beta = \max_{p \in H_1} \mathbb{P}_p(\text{accept } H_0)?$$

- No, because we would get $\beta = 1 - \alpha$.

- As a consequence we calculate β for every value $p \in H_1$ and simply write $\beta(p)$, i.e.

$$\beta(p) = \mathbb{P}_p(\text{accept } H_0) \tag{4}$$

- In our situation we expect $\beta(p)$ to be small if p is very small (close to 0).
- And we expect $\beta(p)$ to be big if p is close to $\frac{1}{2}$.
- The function $\pi(p) = 1 - \beta(p)$ is called **power function** - the higher the value the better.

- Back to the original problem: How to construct a hypothesis test for $H_0 : p \geq \frac{1}{2}$ vs. $H_1 : p < \frac{1}{2}$?
- Why might such a test be of practical relevance?

- The test we are looking for is already implemented in R.
-
```r
#binom.test for testing H0: p>=0.5 versus H1: p<0.5
p <- 0.55
n <- 100
x <- sample(c(0,1),size=n,replace=TRUE,prob=c(1-p,p))
successes <- sum(x)
test <- binom.test(successes,n,p=0.5,alternative="less")
test
```

- yields
-
```
Exact binomial test

data:  successes and n
number of successes = 61, number of trials = 100, p-value = 0.9895
alternative hypothesis: true probability of success is less than 0.5
95 percent confidence interval:
0.0000000 0.6918993
sample estimates:
probability of success
0.61
```

- How can the output be interpreted? Is H_0 rejected or not?
- How is the p-value calculated and what does it tell us?
- We reject H_0 if the **p-value** returned by R is smaller than $\alpha = 0.05$.
- The smaller the p-value the more evidence against H_0.

- Loosely speaking, the p-value is the probability under H_0, to observe 'something at least as extreme as the current value'.
- What does 'something at least as extreme as 61' mean in our case?
- It means that the number of successes X is at most 61.
- In other words:

$$p = \max_{p \in H_0} \mathbb{P}_p(X \leq 61) = \mathbb{P}_{0.5}(X \leq 61) \approx 0.9895$$

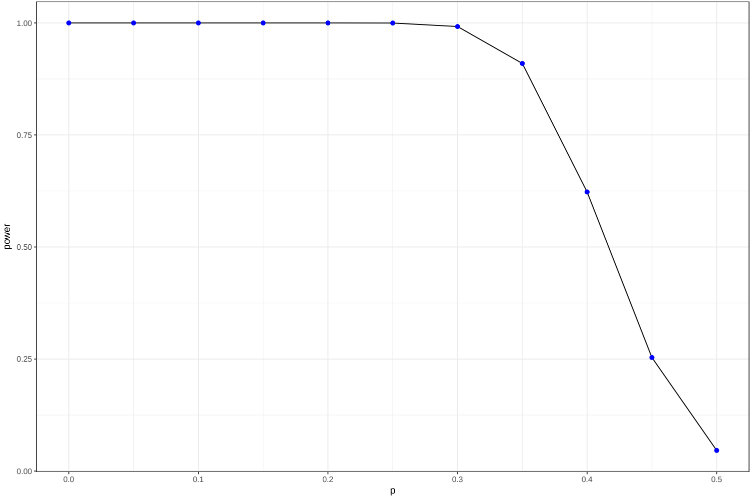
- How can we check if binom.test really does what it should?
- We check by simulations if the type I error is at most 0.05.
- Afterwards we approximate the power function again via simulations.

- We analyze the performance of binom.test via simulations
- ```r
  #assume that H0 holds
  #repeat the above procedure R=10000 times and calculate the portion of false decisions (type I error)
  R <- 10000
  error <- rep(0,R)
  for(i in 1:R){
    p <- 0.6
    n <- 100
    x <- sample(c(0,1),size=n,replace=TRUE,prob=c(1-p,p))
    successes <- sum(x)
    test <- binom.test(successes,n,p=0.5,alternative="less")
    if(test$p.value<0.05){error[i] <- 1}
  }
  mean(error)
  ```
- yields

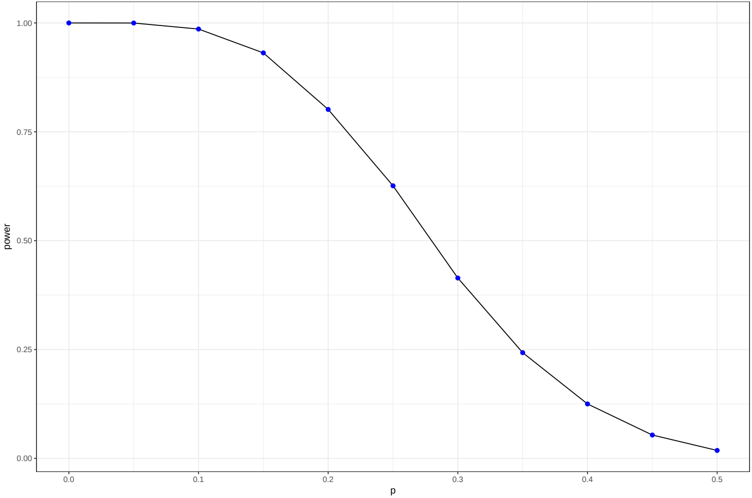
```
[1] 0.0036
```
```

- ```
  #worst case scenario (what is different to before?)
  R <- 10000
  error <- rep(0,R)
  for(i in 1:R){
    p <- 0.5
    n <- 100
    x <- sample(c(0,1),size=n,replace=TRUE,prob=c(1-p,p))
    successes <- sum(x)
    test <- binom.test(successes,n,p=0.5,alternative="less")
    if(test$p.value<0.05){error[i] <- 1}
  }
  mean(error)
  ```
- yields

```
[1] 0.0441
```
```

```r
#@power: choose different values for p in H1 and calculate the power
pgrid <- seq(0,0.5,by=0.05)
power <- rep(0,length(pgrid))
for(j in 1:length(pgrid)){
  print(j)
 R <- 5000
  error <- rep(0,R)
  for(i in 1:R){
   p <- pgrid[j]
   n <- 100
   x <- sample(c(0,1),size=n,replace=TRUE,prob=c(1-p,p))
   successes <- sum(x)
   test <- binom.test(successes,n,p=0.5,alternative="less")
   if(test$p.value>=0.05){error[i] <- 1}        #type II error
  }
 power[j]<-1-mean(error)
}
power
[1] 1.0000 1.0000 1.0000 1.0000 1.0000 0.9998 0.9920 0.9134
     0.6220 0.2532 0.0474
```

Checking the performance of binom.test



```
#@power: same for smaller sample size n
pgrid <- seq(0,0.5,by=0.05)
power <- rep(0,length(pgrid))
for(j in 1:length(pgrid)){
  print(j)
  R <- 5000
  error <- rep(0,R)
  for(i in 1:R){
    p <- pgrid[j]
    n <- 20
    x <- sample(c(0,1),size=n,replace=TRUE,prob=c(1-p,p))
    successes <- sum(x)
    test <- binom.test(successes,n,p=0.5,alternative="less")
    if(test$p.value>=0.05){error[i] <- 1}
  }
  power[j]<-1-mean(error)
}
power
[1] 1.0000 0.9996 0.9876 0.9284 0.8082 0.6130 0.4238 0.2458
     0.1306 0.0548 0.0210
```


1.00
0.75
0.50
0.25
0.00
power
0.0
0.1
0.2
0.3
0.4
0.5
p

Exercise:

- Use binom.test to test the hypothesis $H_0 : p \leq 0.7$ versus $H_1 : p > 0.7$.
- Check that the type I error is at most 0.05 for every $p \in H_0$.
- Calculate/approximate the power function $\pi(p)$ for sample size $n = 100$ via (sufficiently many) simulations.
- Work with different sample sizes, e.g. $n = 10$, $n = 20$, $n = 50$, $n = 100$, $n = 500$, $n = 1000$, and produce a plot of the power function π in each case.
- How can the results be interpreted?

Exercise:

- Use binom.test for testing the hypothesis $H_0 : p = 0.5$ versus $H_1 : p \neq 0.5$.
- Check that the type I error is at most 0.05.
- Calculate/approximate the power function $\pi(p)$ for sample size $n = 100$ via (sufficiently many) simulations.
- Work with different sample sizes, e.g. $n = 10$, $n = 20$, $n = 50$, $n = 100$, $n = 500$, $n = 1000$, and produce a plot of the power function π in each case
- How can the results be interpreted?