Some results on the convergence of (quasi-) copulas

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Abstract

It is shown that pointwise convergence of a sequence $(A_n)_{n\in\mathbb{N}}$ of copulas to a copula A is equivalent (1) to the convergence of the corresponding endographs, and (2) to the convergence of the corresponding upper (or lower) α-levels for all but at most countably many α in $[0,1]$ (all with respect to the Hausdorff metric). Examples are given that show that the countably many exceptions in (2) can not be omitted. It is furthermore shown that the main results also hold on the bigger class of quasi-copulas.

Keywords: Copula, quasi-copula, Hausdorff metric, topological limit, endograph, level set

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1. Introduction

Since the family $\mathcal{C}$ of all two-dimensional copulas is equicontinuous (see [9]) and $[0,1]^2$ is compact pointwise convergence of a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$ to a copula A implies uniform convergence, i.e. $\lim_{n\to\infty} d_\infty(A_n, A) = 0$. Furthermore, using Ascoli-Arzelà theorem (see [10]), the metric space $(\mathcal{C}, d_\infty)$ is easily seen to be compact. Considering the so-called endograph

$$end(A) := \big\{(x,y,t) \in [0,1]^3 : A(x,y) \le t\big\} \tag{1}$$

of a copula A allows to embed $\mathcal{C}$ in the metric space $(\mathcal{K}([0,1]^3), \delta_H)$ of all non-empty compact subsets of $[0,1]^3$ endowed with the Hausdorff metric δ_H. It

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is not difficult to show that the uniform metric and the Hausdorff metric are equivalent on $\mathcal{C}$. Closely related to the endograph of a copula A is the family of its upper and lower level sets, $([A]_\alpha)_{\alpha\in[0,1]}$ and $([A]^\alpha)_{\alpha\in[0,1]}$ respectively, which are defined as

$$[A]_\alpha = \big\{(x, y) \in [0, 1]^2 : A(x, y) \geq \alpha\big\} \tag{2}$$

$$[A]^\alpha = \big\{(x, y) \in [0, 1]^2 : A(x, y) \leq \alpha\big\} \tag{3}$$

for every $\alpha \in [0, 1]$. Having in mind that all these level sets are elements of the metric space $(\mathcal{K}([0, 1]^2), \delta_H)$ the question naturally arises, if pointwise convergence of a sequence $(A_n)_{n\in\mathbb{N}}$ of copulas to a copula A implies convergence of the corresponding upper and lower level sets. We will answer this question and prove that pointwise convergence of $(A_n)_{n\in\mathbb{N}}$ to A is equivalent to each of the following conditions:

(A) There exists a set $\Lambda \subseteq [0, 1]$ of Lebesgue measure 0 such that for all $\alpha \in \Lambda^c$ the equality $\lim_{n\to\infty} \delta_H([A_n]_\alpha, [A]_\alpha) = 0$ holds.

(B) There exists a set $\Gamma \subseteq [0, 1]$ of Lebesgue measure 0 such that for all $\alpha \in \Gamma^c$ the equality $\lim_{n\to\infty} \delta_H([A_n]^\alpha, [A]^\alpha) = 0$ holds.

Furthermore, in the case of pointwise convergence of $(A_n)_{n\in\mathbb{N}}$ to A, it will be shown that the sets Λ, Γ mentioned in (A) and (B) are subsets of the family of discontinuities of the function $\Phi_A : [0, 1] \to \mathcal{K}([0, 1]^2)$, defined by $\Phi_A(\alpha) = [A]_\alpha$, and that the sets Λ, Γ can really be countably infinite. For related, but not directly applicable results (level sets of copulas are not necessarily convex) on upper semicontinuous, [0,1]-valued functions with convex alpha levels see [12]. Since the topology generated by the Hausdorff metric is independent of the concrete chosen metrization of the underlying space (A) and (B) also hold with respect to the myopic and the Fell-topology (see [6] and Section 2). When proving the results only the properties of quasi-copulas $\mathcal{Q}$ (see [9]) are used - consequently all results (except point three in Theorem 6) also hold for quasi-copulas.

2. Notation and preliminaries

Throughout the whole paper $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^d$,

$$B(A, r) := \{y \in \mathbb{R}^d : \exists x \in A \text{ such that } \|x - y\|_2 < r\}$$

the open r-neighbourhood of A, and $\overline{B}(A,r)$ the topological closure of $B(A,r)$ $(r > 0)$. In case of $A = \{z\}$ we will write $B(z,r)$ for the open ball of radius r around z. $\mathcal{K}([0,1]^d)$ denotes the family of all non-empty closed subsets of $[0,1]^d$, $\mathcal{K}_{pc}([0,1]^3)$ the family of all elements in $\mathcal{K}([0,1]^d)$ that are pathwise connected $(d \geq 2)$. The *Hausdorff metric* on $\mathcal{K}([0,1]^d)$ is denoted by δ_H - since no confusion will arise the symbol δ_H will be used for every dimension d. As mentioned in the introduction, the topology generated by the Hausdorff metric is independent of the concrete chosen metrization of the underlying space (see [6]) - we will use the metric ρ_2 induced by $\|\cdot\|_2$ in the definition of δ_H for $d = 2$ and the metric ρ_3, defined by $\rho_3((x_1,y_1,t_1),(x_2,y_2,t_2)) := \max\{\rho_2((x_1,y_1),(x_2,y_2)), |t_2 - t_1|\}$, in the definition of δ_H for $d = 3$.
It is well-known that $(\mathcal{K}([0,1]^d), \delta_H)$ is a compact metric space (see [1]) and it is not difficult to see that in $\mathcal{K}_{pc}([0,1]^d)$ convergence w.r.t. δ_H coincides with the *Painlevé-Kuratowski-(PK for short) convergence* of closed sets (see Proposition 12 in [12]): A sequence $(E_n)_{n\in\mathbb{N}}$ of subsets of an arbitrary metric space (X,ρ) is said to be convergent in the PK-sense if the topological limit inferior ${}^t\liminf_{n\to\infty} E_n$ and the topological limit superior ${}^t\limsup_{n\to\infty} E_n$ coincide, whereby

$$\begin{aligned} {}^t\liminf_{n\to\infty} E_n &:= \{x \in X : \exists (x_n)_{n\in\mathbb{N}} \text{ with } \forall n\, x_n \in E_n \text{ and } \lim_{n\to\infty} x_n = x\} \\ {}^t\limsup_{n\to\infty} E_n &:= \{x \in X : \exists (x_{n_k})_{k\in\mathbb{N}} \text{ with } \forall k\, x_{n_k} \in E_{n_k} \text{ and } \lim_{k\to\infty} x_n = x\} \end{aligned} \tag{4}$$

and $(n_k)_{k\in\mathbb{N}}$ denotes a strictly increasing sequence in $\mathbb{N}$. One additional property of the Hausdorff metric δ_H on a compact metric space (X,ρ) that we will use later is that

$$\lim_{n\to\infty} \delta_H\left(E_n, \overline{\bigcup_{n=1}^{\infty} E_n}\right) = 0 \tag{5}$$

holds for very increasing sequence $(E_n)_{n\in\mathbb{N}}$ of non-empty compact subsets of X ($\overline{E}$ denoting the topological closure of E). For more details on δ_H and PK-convergence see [6], [1], [11].
$\mathcal{C}$ will denote the family of all *two-dimensional copulas*, $\mathcal{Q}$ the family of all two-dimensional quasi-copulas (see [9], [7], [3]). For every copula $A \in \mathcal{C}$ the corresponding doubly stochastic measure will be denoted by μ_A, the family of all these μ_A by $\mathcal{P}_{\mathcal{C}}$. Endowing $\mathcal{C}$ with the uniform distance d_∞ (i.e. $d_\infty(A,B) := \max_{(x,y)\in[0,1]^2} |A(x,y) - B(x,y)|$) yields a compact metric space

$(\mathcal{C}, d_\infty)$. Since copulas are continuous it follows that uniform convergence of a sequence $(A_n)_{n\in\mathbb{N}}$ of copulas to a copula A is equivalent to weak convergence of $(\mu_{A_n})_{n\in\mathbb{N}}$ to $\mu_A \in \mathcal{P}_\mathcal{C}$ (see [2]).
Obviously the *endograph* $end(A)$ of a copula A, defined according to (1), is an element of $\mathcal{K}_{pc}([0,1]^3)$ (in fact, $end(A)$ is even star-shaped). Therefore one can also consider the so-called *endograph metric* D_{end} on $\mathcal{C}$, defined by

$$D_{end}(A,B) := \delta_H(end(A), end(B)) \tag{6}$$

for all $A, B \in \mathcal{C}$. We will see that D_{end} and d_∞ are equivalent metrics.
Given $A \in \mathcal{C}$ we may define the *upper and lower level function* Φ_A, Ψ_A : $[0,1] \to \mathcal{K}_{pc}([0,1]^2)$ by

$$\Phi_A(\alpha) := [A]_\alpha \quad \text{and} \quad \Psi_A(\alpha) := [A]^\alpha \tag{7}$$

for every $\alpha \in [0,1]$. The fact that Φ_A and Ψ_A really map to $\mathcal{K}_{pc}([0,1]^2)$ is a direct consequence of monotonicity and continuity of $A \in \mathcal{C}$. Furthermore it is straightforward to show that Φ_A is strictly decreasing and left-continuous whereas Ψ_A is strictly increasing and right-continuous. Before proving the results mentioned in the Introduction we will take a look to continuity properties of Φ_A and Ψ_A in the next section.

3. Properties of the upper and lower level function of a copula

As first step in proving that Φ_A and Ψ_A have at most countably many discontinuities (w.r.t. δ_H) we will show that the discontinuities of Φ_A and Ψ_A in $(0,1)$ coincide.

Lemma 1. *For every $A \in \mathcal{C}$ the level functions Φ_A and Ψ_A have the same discontinuities in* $(0,1)$.

Proof. It follows directly from (5) that Φ_A has a discontinuity in $\alpha_0 \in (0,1)$ if and only if $[A]_{\alpha_0} \neq \overline{\bigcup_{\alpha>\alpha_0}[A]_\alpha}$ holds, and that Ψ_A has a discontinuity in $\alpha_0 \in (0,1)$ if and only if $[A]^{\alpha_0} \neq \overline{\bigcup_{\alpha<\alpha_0}[A]^\alpha}$ is fulfilled.
Suppose now that $\alpha_0 \in (0,1)$ is a point of discontinuity of Φ_A and set $B := \overline{\bigcup_{\alpha>\alpha_0}[A]_\alpha}$. Then there exists a point $(x,y) \in [0,1)^2$ with $A(x,y) = \alpha_0$ and $\min\big\{\rho_2\big((x,y),(z,w)\big) : (z,w) \in B\big\} = r > 0$. Compactness of B therefore implies the existence of $\Delta > 0$ such that the square $S = [x, x+\Delta] \times [y, y+\Delta]$ fulfills $S \cap B = \emptyset$ and $S \subseteq (0,1)^2$. Hence, by monotonicity of $A \in \mathcal{C}$,

$A(z,w) = \alpha_0$ holds for all $(z,w) \in S$, so $S \subseteq A^{-1}(\{\alpha_0\})$. On the other hand, if there exits a square $S \subseteq (0,1)^2 \cap A^{-1}(\{\alpha_0\})$ with non-empty interior then we have $[A]_{\alpha_0} \neq \overline{\bigcup_{\alpha>\alpha_0}[A]_\alpha}$, so α_0 is a discontinuity point of Φ_A.
Following the same line of argumentation it can be shown that Ψ_A has a discontinuity in $\alpha_0 \in (0,1)$ if and only if there exits a square $S \subseteq (0,1)^2 \cap A^{-1}(\{\alpha_0\})$ with non-empty interior. ■

Based on Lemma 1 it suffices to analyze the discontinuities of Φ_A, which can be done by using the so-called *radius-vector function* R_A of the upper level sets of A (see [6]). R_A is defined as

$$R_A(\alpha,\varphi) := \max\left\{t \geq 0 : (1,1) + t(\cos(\varphi),\sin(\varphi)) \in [A]_\alpha\right\} \tag{8}$$

for every $\alpha \in [0,1]$ and $\varphi \in [\pi, 3\pi/2]$. Based on R_A we set

$$(x_\alpha(\varphi), y_\alpha(\varphi)) = (1,1) + R_A(\alpha,\varphi)(\cos(\varphi),\sin(\varphi)) \tag{9}$$

for every $\alpha \in [0,1]$ and $\varphi \in [\pi, 3\pi/2]$. Using monotonicity it is straightforward to see that $A\big((x_\alpha(\varphi), y_\alpha(\varphi))\big) = \alpha$ holds for all $\alpha \in [0,1]$ and $\varphi \in [\pi, 3\pi/2]$. Some further properties of R_A are collected in the following lemma.

Lemma 2. *For every $A \in \mathcal{C}$ the radius-vector function R_A has the following properties:*

(a) For every fixed $\alpha \in [0,1]$ the function $\varphi \mapsto R_A(\alpha,\varphi)$ is continuous.

(b) For every fixed $\varphi \in [\pi, 3\pi/2]$ the function $\alpha \mapsto R_A(\alpha,\varphi)$ is left-continuous and strictly decreasing.

(c) The function $\varphi \mapsto x_\alpha(\varphi)$ is monotonically non-decreasing and continuous, $\varphi \mapsto y_\alpha(\varphi)$ is monotonically non-increasing and continuous.

(d) For every compact subinterval $I \subset (\pi, 3\pi/2)$ there exists a constant L such that all functions $\varphi \mapsto R_A(\alpha,\varphi)$ are Lipschitz continuous with common Lipschitz constant L on I.

(e) If $(\alpha_n)_{n\in\mathbb{N}}$ is monotonically decreasing with limit $\alpha \in [0,1)$ and $R_A(\alpha_n,\varphi)$ converges to $R_A(\alpha,\varphi)$ for every φ then $\lim_{n\to\infty} \delta_H([A]_{\alpha_n}, [A]_\alpha) = 0$ holds.

PROOF. Since $R_A(\alpha,\varphi) = \rho_2\big((x_\alpha(\varphi), y_\alpha(\varphi)), (1,1)\big)$ holds (a) is a direct consequence of (c).

The fact that $\alpha \mapsto R_A(\alpha, \varphi)$ is monotonically non-increasing is obvious. Furthermore $R_A(\alpha, \varphi) = R_A(\beta, \varphi)$ for $\alpha \leq \beta$ implies

$$\alpha = A\big((x_\alpha(\varphi), y_\alpha(\varphi)\big) = A\big((x_\beta(\varphi), y_\beta(\varphi)\big) = \beta,$$

showing that $\alpha \mapsto R_A(\alpha, \varphi)$ is strictly decreasing. If $(\alpha_n)_{n\in\mathbb{N}}$ is monotonically increasing to $\alpha \in (0, 1]$ then $t := \lim_{n\to\infty} R_A(\alpha_n, \varphi) \geq R_A(\alpha, \varphi)$ holds. Set $t_n := R_A(\alpha_n, \varphi)$ for every $n \in \mathbb{N}$. Because of $\lim_{n\to\infty} \delta_H(A_{\alpha_n}, A_\alpha) = 0$ using (4) it follows that

$$(1, 1)+t(\cos(\varphi), \sin(\varphi)) = \lim_{n\to\infty}(1, 1)+t_n(\cos(\varphi), \sin(\varphi)) \in {}^t\liminf_{n\to\infty} A_{\alpha_n} = A_\alpha,$$

so $t \leq R_A(\alpha, \varphi)$. This completes the proof of point (b). Suppose that $\varphi < \psi$ and that $\varphi, \psi \in [\pi, 3\pi/2]$. Using the fact that as copula A is coordinate-wise monotonic (see [9]) it is straightforward to see that $x_\alpha(\varphi) < x_\alpha(\psi), y_\alpha(\psi) < y_\alpha(\varphi)$ cannot hold - we would find a rectangle with non-empty interior having $(x_\alpha(\varphi), y_\alpha(\varphi))$ as upper right corner on which A only assumes the value α, which contradicts the construction of $(x_\alpha(\varphi), y_\alpha(\varphi))$. Analogously $x_\alpha(\varphi) > x_\alpha(\psi), y_\alpha(\psi) > y_\alpha(\varphi)$ cannot hold. Consequently $x_\alpha(\varphi) \geq x_\alpha(\psi), y_\alpha(\psi) \leq y_\alpha(\varphi)$ follows, which proves the stated monotonicity properties. To show continuity we can proceed as follows: Suppose that $\varphi < \psi$ and that $\varphi, \psi \in (\pi, 3\pi/2)$ holds. Using some trigonometry one gets

$$\begin{aligned} |x_\alpha(\varphi) - x_\alpha(\psi)| &\leq |\tan(3\pi/2 - \psi) - \tan(3\pi/2 - \varphi)| \\ |y_\alpha(\varphi) - y_\alpha(\psi)| &\leq |\cot(3\pi/2 - \psi) - \cot(3\pi/2 - \varphi)| \end{aligned} \tag{10}$$

which shows Lipschitz continuity on every compact subinterval $I \subset (\pi, 3\pi/2)$. If $(\varphi_n)_{n\in\mathbb{N}}$ is a monotonically decreasing sequence in $(\pi, 3\pi/2]$ with limit π then clearly $\lim_{n\to\infty} y_\alpha(\varphi_n) = 1$, so

$$\lim_{n\to\infty} A\big((x_\alpha(\varphi_n), y_\alpha(\varphi_n)\big) = A\big(\lim_{n\to\infty} x_\alpha(\varphi_n), 1\big) = \alpha,$$

and therefore $\lim_{n\to\infty} x_\alpha(\varphi_n) = \alpha$ follows. Right continuity at $\varphi = 3\pi/2$ can be shown in the same way. Assertion (d) is a direct consequence of (10). Finally, (e) follows from the fact that

$$\delta_H([A]_{\alpha_n}, [A]_\alpha) \leq \max\big\{|R_A(\alpha_n, \varphi) - R_A(\alpha, \varphi)| : \varphi \in [\pi, 3\pi/2]\big\} \tag{11}$$

in combination with Dini's theorem on the monotone convergence of continuous functions on a compact metric space (see [5]). ■

Remark 1. Again with some trigonometry one can show that for every $\varphi \in (\pi, 3\pi/2)$ and $\alpha, \beta \in [0,1]$ the following inequality holds:

$$|R_A(\alpha,\varphi) - R_A(\beta,\varphi)| \le \max\left\{\frac{\delta_H([A]_\alpha,[A]_\beta)}{\sin(3\pi/2-\varphi)}, \frac{\delta_H([A]_\alpha,[A]_\beta)}{\cos(3\pi/2-\varphi)}\right\} \tag{12}$$

Using (12) and Dini's theorem it is easy to see that for every monotonically decreasing sequence $(\alpha_n)_{n\in\mathbb{N}}$ with limit $\alpha \in [0,1)$ $\lim_{n\to\infty} \delta_H([A]_{\alpha_n},[A]_\alpha) = 0$ implies

$$\lim_{n\to\infty} \max\big\{|R_A(\alpha_n,\varphi) - R_A(\alpha,\varphi)| : \varphi \in [\pi, 3\pi/2]\big\} = 0$$

Since this result will not be needed in the sequel the proof is omitted.

Theorem 3. *For every $A \in \mathcal{C}$ the upper level function Φ_A has at most countably many discontinuities. The same holds for the lower level function Ψ_A.*

Proof. For every $\varphi \in [\pi, 3\pi/2]$ let D_φ denote set of all discontinuities of the function $f_\varphi : \alpha \mapsto R_A(\alpha,\varphi)$. Since f_φ is decreasing and bounded D_φ is at most countably infinite. Set $D := \bigcup_{\varphi\in[\pi,3\pi/2]\cap\mathbb{Q}} D_\varphi$, then D is as countable union of countable sets itself at most countably infinite. We will show that for arbitrary $\psi \in (\pi, 3\pi/2)$ every discontinuity of f_ψ is contained in D.
Suppose that $\psi \in (\pi, 3\pi/2)$ and let $\alpha_0 \in D_\psi$, then $a := f_\psi(\alpha_0) - f_\psi(\alpha_0+) > 0$. Choose $r > 0$ sufficiently small so that $I := [\psi - r, \psi + r] \subseteq (\pi, 3\pi/2)$. Property four in Lemma 2 implies the existence of a constant $L > 0$ such that for every $\alpha \in [0,1]$ the function $\varphi \mapsto R_A(\alpha,\varphi)$ is Lipschitz continuous on I with Lipschitz constant L. Consequently there exists $\delta \in (0,r)$ such that

$$\sup_{\varphi\in[\psi-\delta,\psi+\delta]} |R_A(\alpha,\varphi) - R_A(\alpha,\psi)| \le \frac{a}{3}$$

holds for every $\alpha \in [0,1]$. Hence, choosing $\varphi \in [\psi-\delta,\psi+\delta]\cap\mathbb{Q}$ it follows that both $|f_\varphi(\alpha_0) - f_\psi(\alpha_0)| \le a/3$ and $|f_\varphi(\alpha_0+) - f_\psi(\alpha_0+)| \le a/3$ is fulfilled. Consequently $f_\varphi(\alpha_0) - f_\varphi(\alpha_0+) \ge a/3$ and $\alpha_0 \in D$.
For every $\alpha_0 \in D^c$ we have continuity of every function f_φ, $\varphi \in [\pi, 3\pi/2)]$, in α_0. Applying property five of Lemma 2 and using left-continuity of Φ_A therefore shows that every $\alpha_0 \in D^c$ is also a continuity point of Φ_A. ■

The next example shows the existence of a copula $A^\star \in \mathcal{C}$ for which the function $\Phi_{A^\star}$ has infinitely many discontinuities.

Example 1. We will use the construction of copulas with fractal support described in [4] which is based on special *Iterated Function Systems* (IFS for short) coming from so-called transformation matrices. Defining $\mathbf{T}$ as

$$\mathbf{T} = \begin{pmatrix} 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \\ 1/3 & 0 & 0 \end{pmatrix}$$

corresponds to the IFS $\{T_1, T_2, T_3\}$, whereby the contractions $T_i, i = 1, 2, 3$, are defined by

$$T_1(x,y) = \frac{1}{3}(x,y)\,,\; T_2(x,y) = \frac{1}{3}(x,y) + \left(\frac{1}{3},\frac{2}{3}\right)\,,\; T_3(x,y) = \frac{1}{3}(x,y) + \left(\frac{2}{3},\frac{1}{3}\right)$$

for all $(x, y) \in [0, 1]^2$ (see [1]). The IFS $\{T_1, T_2, T_3\}$ induces an operator $V : \mathcal{P}([0, 1]^2) \mapsto \mathcal{P}([0, 1]^2)$, defined by

$$V(\mu) := \frac{1}{3}\sum_{i=1}^{3} \mu^{T_i}, \tag{13}$$

whereby $\mathcal{P}([0, 1]^2)$ denotes the family of all probability measures on the Lebesgue sigma field of $[0, 1]^2$ and μ^T is the measure induced by the transformation T. Let ρ denote a metrization (for instance the Hutchinson metric, see [1]) of the weak convergence on $\mathcal{P}([0, 1]^2)$. Then $(\mathcal{P}([0, 1]^2), \rho)$ is a compact metric space and it can be shown that V is a contraction, so by the Banach fixed point theorem there exists exactly one invariant measure $\mu^\star$ that is globally attractive, i.e. for every $\mu \in \mathcal{P}([0, 1]^2)$ we have $\lim_{n\to\infty} \rho(V^n(\mu), \mu^\star) = 0$ (again see [1]). It is easy to verify that $V(\mu_A) \in \mathcal{P}_\mathcal{C}$ holds for every $A \in \mathcal{C}$ (see [4]), so the operator V maps copulas to copulas, and that $\mathcal{P}_\mathcal{C}$ is a closed subset of $(\mathcal{P}([0, 1]^2), \rho)$. Consequently there exists an unique V-invariant measure $\mu_{A^\star}$ that is globally attractive. Figure 1 shows the (support of the) mass distribution of $V(\mu_\Pi)$ and $V^2(\mu_\Pi)$, whereby Π denotes the product copula.

It follows directly from the construction that $\mu_{A^\star}$ is a singular measure and that

$$\mu_{A^\star}\left(\left[\frac{1}{3^n},\frac{2}{3^n}\right]^2\right) = 0 \quad \text{for every } n \geq 1.$$

Consequently (see the proof of Lemma 1) $\alpha_n := A^\star(3^{-n}, 3^{-n}) = 3^{-n}$ is a discontinuity point of the upper level function $\Phi_{A^\star}$ for every $n \geq 1$.

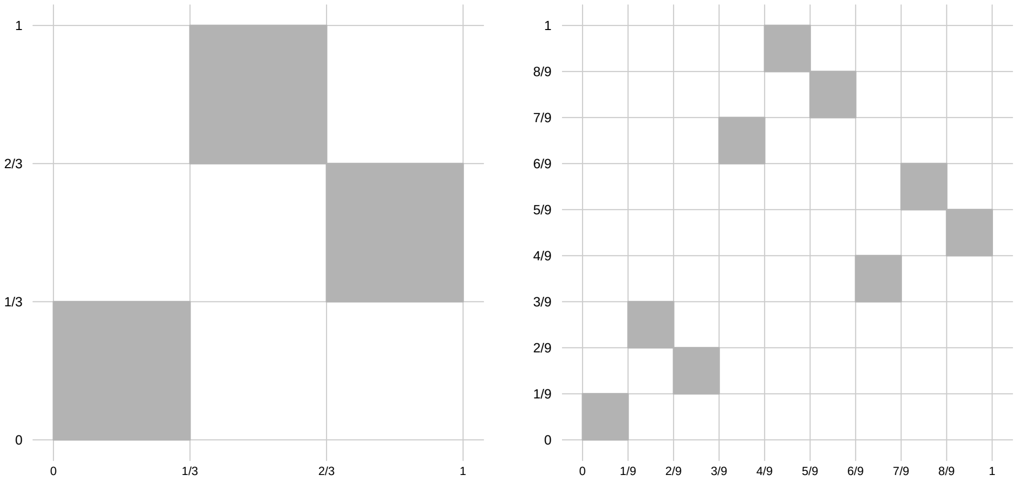

Figure 1: (Support of) $V(\mu_\Pi)$ and $V^2(\mu_\Pi)$

Remark 2. The IFS-construction of copulas can also be used to show that there exists no constant L such that

$$|R_A(\alpha,\varphi) - R_A(\beta,\varphi)| \leq L\,\delta_H([A]_\alpha,[A]_\beta)$$

holds simultaneously for all $\varphi \in [\pi, 3\pi/2]$ and $\alpha, \beta \in [0,1]$ (see inequality (12)). One can, for instance, proceed as follows: Start with the transformation matrix

$$\mathbf{T}_k = \begin{pmatrix} 0 & 1/k & 0 & \dots & 0 \\ 0 & 0 & 1/k & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1/k \\ 1/k & 0 & 0 & \dots & 0 \end{pmatrix},$$

consider the corresponding IFS $\{T_1, T_2, \dots, T_k\}$ and define the operator $V_k : \mathcal{P}([0,1]^2) \mapsto \mathcal{P}([0,1]^2)$ (mapping $\mathcal{P}_\mathcal{C}$ into itself) as

$$V_k(\mu) := \frac{1}{k}\sum_{i=1}^{k} \mu^{T_i}. \tag{14}$$

Denote by M the minimum-copula and consider the copula A_k such that $\mu_{A_k} = V_k(\mu_M)$, define $\alpha_k := 1/k + 1/k^3$, $\beta_k := 1/k + 1/k^2$, $(x_k, y_k) :=$

$(1/k + 1/k^{3/2}, (k-1)/k + 1/k^{3/2})$, and verify that

$$\delta_H\big([A_k]_{\alpha_k}, [A_k]_{\beta_k}\big) = \frac{\sqrt{2}(\sqrt{k}-1)}{k^2}$$

as well as

$$\big|R_{A_k}(\alpha_k, \varphi_k) - R_A(\beta_k, \varphi_k)\big| \geq \frac{1}{k}$$

holds for $k \geq 4$. Thereby φ_k denotes the angle in the interval $[\pi, 3\pi/2]$ such that $\tan(\varphi_k + \pi) = \frac{\sqrt{k}-1}{\sqrt{k}(k-1)}$ holds. Figure 2 denotes the support of $V_k(\mu_\Pi)$.

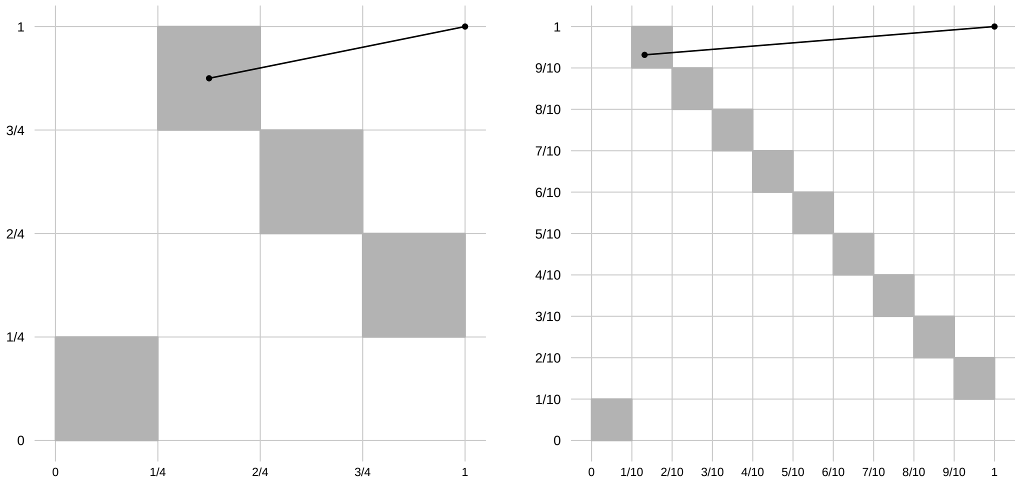

Figure 2: (Support of) $V_k(\mu_\Pi)$ and (x_k, y_k) for $k = 4$ and $k = 10$

Remark 3. Given $A \in \mathcal{C}$ the *Kendall distribution function* $K_A : [0,1] \to [0,1]$ *of* A is defined by (see [8], [9])

$$K_A(t) := \mu_A\big([A]^t\big)$$

for every $t \in [0,1]$. Since K_A is also defined in terms of (upper) level sets the question may arise whether all discontinuity points of Φ_A are also discontinuity points of K_A and/or vice versa. Using, for instance, again radius-vector functions it is not difficult to show that K_A is continuous for absolutely continuous copulas. Since level functions of absolutely continuous copulas may have discontinuities (e.g. $V(\mu_\Pi)$ with V as in Example 1) this implies that a discontinuity point of Φ_A is not necessarily a discontinuity point of K_A, so the first part of the conjecture is wrong. The second part is wrong too. Consider, for instance, the copula $A = 1/2\,(W+B)$, whereby W denotes the lower Fréchet-Hoeffding bound (see [9]) and B denotes the copula corresponding to the uniform distribution on $[0,1/2]^2 \cup [1/2,1]^2$. Then $\mu_A\big(A^{-1}(\{1/2\})\big) = 1/4$, so $t_0 = 1/2$ is a discontinuity point of K_A. Nevertheless Φ_A is continuous at t_0.

4. Main results

Lemma 4. *Suppose that $(A_n)_{n\in\mathbb{N}}$ is a sequence of copulas that converges pointwise to $A \in \mathcal{C}$. If $\alpha \in (0,1)$ is a continuity point of Φ_A, then*

$$\lim_{n\to\infty} \delta_H\big([A_n]_\alpha, [A]_\alpha\big) = \lim_{n\to\infty} \delta_H\big([A_n]^\alpha, [A]^\alpha\big) = 0$$

holds.

Proof. Let $\varepsilon > 0$. Then there exists $\delta \in (0,\varepsilon)$ such that $\delta_H([A]_\alpha, [A]_\beta) \leq \varepsilon$ whenever $|\alpha - \beta| \leq \delta$. Since pointwise convergence of $(A_n)_{n\in\mathbb{N}}$ to A implies uniform convergence there exists an index $n_0 \in \mathbb{N}$ with $d_\infty(A_n, A) \leq \delta$ for every $n \geq n_0$. If $(x,y) \in [A]_\alpha$ then we can find $(z,w) \in [A]_{\alpha+\delta}$ fulfilling $\rho_2((x,y),(z,w)) \leq \varepsilon$. Consequently $A_n(z,w) \geq \alpha$ and $(z,w) \in [A_n]_\alpha$ holds for all $n \geq n_0$. Since $(x,y) \in [A]_\alpha$ was arbitrary $[A]_\alpha \subseteq \overline{B}([A_n]_\alpha, \varepsilon)$ is fulfilled for every $n \geq n_0$. On the other hand, if $(x,y) \in [A_n]_\alpha$ and $n \geq n_0$ then $A(x,y) \geq \alpha - \delta$ and $(x,y) \in [A]_{\alpha-\delta}$ follows. We can find $(z,w) \in [A]_\alpha$ such that $\rho_2((x,y),(z,w)) \leq \varepsilon$ holds. Since $(x,y) \in [A_n]_\alpha$ was arbitrary we get $[A_n]_\alpha \subset \overline{B}([A]_\alpha, \varepsilon)$. Altogether this shows that $\delta_H([A_n]_\alpha, [A]_\alpha) \leq \varepsilon$ holds for every $n \geq n_0$. Convergence of the lower α-level sets can be proved analogously. ■

In the following example copulas $A, A_1, A_2 \ldots$ are constructed such that $\delta_H([A_n]_\alpha, [A]_\alpha) \not\to 0$ holds for infinitely many $\alpha \in (0,1)$ although at the same time $\lim_{n\to\infty} d_\infty(A_n, A) = 0$.

Example 2. Let $A^\star$ be the invariant copula from Example 1 and define W as the lower Fréchet-Hoeffding bound (see [9]), i.e. $W(x,y) := \max\{x+y-1, 0\}$ for all $(x,y) \in [0,1]^2$. Set $A_n := (1-1/n)A^\star + 1/n\,W$, then obviously $A_n(x,y)$ converges monotonically to $A(x,y)$ for every $(x,y) \in [0,1]^2$. For every $k \in \mathbb{N}$ we get $[A^\star]_{3^{-k}} = [3^{-k}, 1]^2$. On the other hand, setting $(x_k, y_k) := (2^{-1}3^{-k+1}, 2^{-1}3^{-k+1})$ yields $(x_k, y_k) \in [A^\star]_{3^{-k}}$ and

$$A_n(x_k, y_k) = \frac{n-1}{n} A^\star(x_k, y_k) = \frac{n-1}{n}\frac{1}{3^k} < \frac{1}{3^k}$$

for all $n \in \mathbb{N}$. Therefore, using monotonicity,

$$\delta_H\big([A_n]_{3^{-k}}, [A^\star]_{3^{-k}}\big) \geq \frac{1}{3^{k-1}}\Big(\frac{1}{2} - \frac{1}{3}\Big) = \frac{1}{3^{k-1}}\Big(\frac{1}{6}\Big)$$

holds for every n, which shows that $\delta_H([A_n]_{3^{-k}}, [A]_{3^{-k}}) \not\to 0$.

Lemma 5. *Suppose that $A, A_1, A_2, \dots$ are copulas and that one of the following two conditions are fulfilled:*

(A) There exists a set $\Lambda \subseteq [0,1]$ of Lebesgue measure 0 such that for all $\alpha \in \Lambda^c$ the equality $\lim_{n\to\infty} \delta_H([A_n]_\alpha, [A]_\alpha) = 0$ holds.

(B) There exists a set $\Gamma \subseteq [0,1]$ of Lebesgue measure 0 such that for all $\alpha \in \Gamma^c$ the equality $\lim_{n\to\infty} \delta_H([A_n]^\alpha, [A]^\alpha) = 0$ holds.

Then $\lim_{n\to\infty} d_\infty(A_n, A) = 0$ follows.

Proof. Suppose that (A) holds. Fix $(x,y) \in [0,1]^2$ and set $\alpha = A(x,y)$. If $\alpha > 0$ and $k \in \mathbb{N}$ then there exists $\beta \in (\alpha - 1/k, \alpha] \cap \Lambda^c$ since Λ^c is dense in $[0,1]$. Using $(x,y) \in [A]_\alpha \subseteq [A]_\beta$, $\lim_{n\to\infty} \delta_H([A_n]_\beta, [A]_\beta) = 0$, and (4) therefore shows the existence of a sequence $((x_n, y_n))_{n\in\mathbb{N}}$ converging to (x,y) and fulfilling $(x_n, y_n) \in [A_n]_\beta$ for every $n \in \mathbb{N}$. Lipschitz continuity (see [9]) implies

$$A_n(x,y) \geq A_n(x_n, y_n) - \sqrt{2}\,\rho_2\big((x_n, y_n), (x,y)\big) \geq \beta - \sqrt{2}\rho_2\big((x_n, y_n), (x,y)\big)$$

from which $\liminf_{n\to\infty} A_n(x,y) \geq \beta$ follows. Since k was arbitrary we get $\liminf_{n\to\infty} A_n(x,y) \geq \alpha$. In case of $\alpha = 0$ this inequality is clearly valid. Assume that there exists $r > 0$ such that $A_n(x,y) \geq \alpha + r$ holds for infinitely many $n \in \mathbb{N}$. For every $\beta \in (\alpha, \alpha + r) \cap \Lambda^c$ we get $(x,y) \in [A_n]_\beta$ infinitely often, hence $(x,y) \in {}^t\limsup_{n\to\infty}[A_n]_\beta = [A]_\beta$ follows, which contradicts $A(x,y) = \alpha$. Consequently $\limsup_{n\to\infty} A_n(x,y) \leq \alpha$, and $\lim_{n\to\infty} A_n(x,y) = \alpha$ holds. If (B) holds pointwise convergence can be proved analogously. ■

Altogether we get the following theorem:

Theorem 6. *Suppose that $A, A_1, A_2, \dots$ are copulas. Then the following conditions are equivalent:*

(a) $\lim_{n\to\infty} \big|A_n(x,y) - A(x,y)\big| = 0$ *for every* $(x,y) \in [0,1]^2$.

(b) $\lim_{n\to\infty} d_\infty(A_n, A) = 0$.

(c) The sequence $(\mu_{A_n})_{n\in\mathbb{N}}$ *converges weakly to* μ_A.

(d) $\lim_{n\to\infty} D_{end}(A_n, A) = 0$.

(e) There exists a set $\Lambda \subseteq [0,1]$ *of Lebesgue measure 0 such that for all* $\alpha \in \Lambda^c$ *the equality* $\lim_{n\to\infty} \delta_H([A_n]_\alpha, [A]_\alpha) = 0$ *holds.*

(f) There exists a set $\Gamma \subseteq [0,1]$ *of Lebesgue measure 0 such that for all* $\alpha \in \Gamma^c$ *the equality* $\lim_{n\to\infty} \delta_H([A_n]^\alpha, [A]^\alpha) = 0$ *holds.*

Proof. Since the complement of every subset of $[0,1]$ with Lebesgue measure zero is dense in $[0,1]$ the only equivalence left to prove is (d). It suffices to show that the following inequality holds for all $A, B \in \mathcal{C}$:

$$D_{end}(A,B) \le d_\infty(A,B) \le (1+\sqrt{2})D_{end}(A,B) \tag{15}$$

Since for $(x,y,t) \in end(A)$ there exists $(x,y,s) \in end(B)$ such that $|t-s| \le |A(x,y) - B(x,y)| \le d_\infty(A,B)$, the first part of (15) is obvious.
To prove the second part fix $(x,y) \in [0,1]^2$ and set $\Delta := D_{end}(A,B)$. Assume that $A(x,y) \ge B(x,y)$. Then, because of $(x,y,A(x,y)) \in end(A)$, there exists $(z,w,s) \in end(B)$ such that $\rho_2((x,y)(z,w)), |A(x,y) - s| \le \Delta$. Using Lipschitz continuity $|B(x,y) - B(z,w)| \le \sqrt{2}\rho_2((x,y)(z,w)) \le \sqrt{2}\Delta$, and therefore $B(x,y) \ge B(z,w) - \sqrt{2}\Delta \ge A(x,y) - (1+\sqrt{2})\Delta$ follows. This shows that in this case $0 \le A(x,y) - B(x,y) \le (1+\sqrt{2})\Delta$ holds, from which the second part of (15) follows. ■

Remark 4. Since the only properties of copulas that were used throughout the paper are Lipschitz-continuity, coordinate-wise monotonicity and fulfillment of the boundary conditions all assertions and results formulated before for the class $\mathcal{C}$ (except point three in Theorem 6) also hold for the bigger class of quasi copulas $\mathcal{Q}$.

5. Conclusion and future work

Some conditions equivalent to the pointwise convergence of (quasi-) copulas have been proved. In particular, it was shown that pointwise convergence of copulas is equivalent to the convergence of almost all upper (or lower) level sets and to the convergence of the corresponding endographs. As next steps it seems reasonable to study further properties of the radius-vector function of (quasi-) copulas (which was mainly used as a vehicle to show that the level-functions cannot have more than countably many discontinuities) and to extend the stated results to the family of all d-dimensional (quasi-) copulas for $d \geq 3$.

[1] M.F. Barnsley: *Fractals everywhere*, Academic Press, Cambridge, 1993

[2] P. Billingsley: *Convergence of Probability Measures*, John Wiley & Sons, New York (1968)

[3] F. Durante, A. Kolesárová, R. Mesiar, C. Sempi, Semilinear copulas, *Fuzzy Sets Syst.* **159** Issue 1 (2008) 63-76

[4] G.A. Fredricks, R.B. Nelsen, J.A. Rodríguez-Lallena, Copulas with fractal supports, *Insur. Math. Econ.* **37** (2005) 42-48

[5] E. Hewitt, K. Stromberg: *Real and Abstract Analysis*, Springer Verlag, Berlin Heidelberg, 1965

[6] I. Molchanov: *Theory of Random Sets*, Springer, London, 2005

[7] R.B. Nelsen, J.J. Quesada-Molina, J.A. Rodríguez-Lallena, M. Úbeda-Flores, Quasi-copulas and signed measures, *Fuzzy Sets Syst.* **161**, Issue 17 (2010) 2328-2336

[8] R.B. Nelsen, J.J. Quesada-Molina, J.A. Rodríguez-Lallena, M. Úbeda-Flores, Kendall distribution functions, *Stat. Probabil. Lett.* **65** (2003) 263-268

[9] R.B. Nelsen: *An Introduction to Copulas*, Springer, New York, 2006

[10] W. Rudin: *Real and Complex Analysis*, McGraw-Hill International Editions, Singapore, 1987

[11] R. Schneider: *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge University Press, 1993

[12] W. Trutschnig: Characterization of the sendograph-convergence of fuzzy sets by means of their L_p- and levelwise convergence, *Fuzzy Sets Syst.* **161**, Issue 8 (2010) 1064-1077