Singularity aspects of Archimedean copulas

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Abstract

Calculating Markov kernels of two-dimensional Archimedean copulas allows for very simple and elegant alternative derivations of various important formulas including Kendall's distribution function and the measures of the level curves. More importantly, using Markov kernels we prove the existence of singular Archimedean copulas A_φ with full support of the following two types: (i) All conditional distribution functions $y \mapsto F_x^{A_\varphi}(y)$ are discrete and strictly increasing; (ii) all conditional distribution functions $y \mapsto F_x^{A_\varphi}(y)$ are continuous, strictly increasing and have derivative zero almost everywhere. The results show that despite of their simple analytic form Archimedean copulas can exhibit surprisingly singular behavior.

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1. Introduction

Being the link between multivariate distribution functions and their marginals copulas are a fundamental tool in dependence modeling. Archimedean copulas form an important subclass of copulas which has been successfully applied in various fields like finance and hydrology (see, for instance, [4, 13,

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14] and the references therein), mainly due to their simple analytic form. In fact, every Archimedean copula is fully characterized in terms of a single convex, strictly decreasing function $\varphi : [0,1] \to [0,\infty]$ called the generator. In the Archimedean setting important quantities can be calculated explicitly and expressed as simple formulas involving only the generator (again see [14]). It is also well known that (weak) convergence of Archimedean copulas can easily be characterized by properties of the corresponding generators (see [2]).

In the current paper we concentrate on singularity aspects of Archimedean copulas and prove that, despite their simple analytic form, they may exhibit very singular behaviour when it comes to the distribution of mass. More precisely, we prove the existence two different types of singular Archimedean copulas A_φ with full support: (i) All conditional distribution functions $y \mapsto F_x^{A_\varphi}(y)$ are discrete and strictly increasing; (ii) all conditional distribution functions $y \mapsto F_x^{A_\varphi}(y)$ are continuous, strictly increasing and have derivative zero almost everywhere. Copulas with property (ii) have already been constructed in [20] with the help of Iterated Function Systems with Probabilities and Ergodic Theory - at first sight it seems surprising that such a peculiar mass distribution is also possible for Archimedean copulas.

The rest of the paper is organized as follows: Section 2 gathers some preliminaries and notations that will be used throughout the paper. Section 3 calculates Markov kernels (regular conditional distributions) of Archimedean copulas and demonstrates that well-known results/formulas for Archimedean copulas are straightforwardly derivable when working with Markov kernels. Finally, Section 4 contains the construction of the afore-mentioned types of singular Archimedean copulas with full support.

2. Notation and preliminaries

In the sequel $\mathcal{C}$ will denote the family of all two-dimensional *copulas*, $\mathcal{P}_\mathcal{C}$ the family of all *doubly stochastic measures*, see [3, 14, 17]. For every $A \in \mathcal{C}$ the corresponding doubly stochastic measure will be denoted by μ_A. Following [14] a function $\varphi : [0,1] \to [0,\infty]$ is called *generator* if φ is convex, strictly decreasing and fulfills $\varphi(1) = 0$. A generator φ is called *strict* if $\varphi(0) = \infty$ holds. In case of $\varphi(0) < \infty$ we will refer to φ as *non-strict.* Every (strict or non-strict) generator φ induces a symmetric copula A_φ via

$$A_\varphi(x,y) = \varphi^{[-1]}\big(\varphi(x) + \varphi(y)\big), \qquad x, y \in [0,1],$$

to which we will refer as the (strict or non-strict) Archimedean copula induced by φ. Thereby the pseudo-inverse $\varphi^{[-1]} : [0,\infty] \to [0,1]$ of φ is defined by

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t) & \text{if } t \in [0, \varphi(0)) \\ 0 & \text{if } t \geq \varphi(0). \end{cases}$$

If φ is strict then $\varphi^{[-1]}$ coincides with the standard inverse and it is straightforward to verify that, for given $x \in (0,1]$ the function $y \mapsto A_\varphi(x,y)$ is strictly increasing.

Furthermore, it is well known (see again [14]) that for Archimedean copulas the level set $L_t := \{(x,y) \in [0,1]^2 : A_\varphi(x,y) = t\}$ is a convex curve for every $t \in (0,1]$. For $t = 0$ we get $L_0 = (\{0\} \times [0,1]) \cup ([0,1] \times \{0\})$ if φ is strict whereas L_0 has positive area if φ is non-strict. Defining $f^t : [t,1] \to [0,1]$ by

$$f^t(x) := \varphi^{-1}(\varphi(t) - \varphi(x)) \tag{1}$$

we obviously have

$$\Gamma(f^t) := \big\{(x, f^t(x)) : x \in [t,1]\big\} = L_t \tag{2}$$

for every $t \in (0,1]$, i.e. the graph of f^t coincides with the level curve L_t. Additionally, if φ is non-strict $L_0 = \{(x,y) \in [0,1]^2 : y \leq f^0(x)\}$ holds.

In the sequel $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-field in $\mathbb{R}$, λ and λ_2 the Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^2$ respectively. A *Markov kernel* from $\mathbb{R}$ to $\mathcal{B}(\mathbb{R})$ is a mapping $K : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0,1]$ such that $x \mapsto K(x,B)$ is measurable for every fixed $B \in \mathcal{B}(\mathbb{R})$ and $B \mapsto K(x,B)$ is a probability measure for every fixed $x \in \mathbb{R}$. If we only have $K(x,\mathbb{R}) \in [0,1]$ then $K(\cdot,\cdot)$ will be called *substochastic* kernel. Suppose that X, Y are real-valued random variables on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$, then a Markov kernel $K : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0,1]$ is called a *regular conditional distribution of* Y *given* X if for every $B \in \mathcal{B}(\mathbb{R})$

$$K(X(\omega), B) = \mathbb{E}(\mathbf{1}_B \circ Y | X)(\omega) \tag{3}$$

holds $\mathcal{P}$-a.e. It is well known that for each pair (X,Y) of real-valued random variables a regular conditional distribution $K(\cdot,\cdot)$ of Y given X exists, that $K(\cdot,\cdot)$ is unique $\mathcal{P}^X$-a.s. (i.e. unique for $\mathcal{P}^X$-almost all $x \in \mathbb{R}$) and that $K(\cdot,\cdot)$ only depends on $\mathcal{P}^{X \otimes Y}$. Hence, given $A \in \mathcal{C}$ we will denote (a version of) the regular conditional distribution of Y given X by $K_A(\cdot,\cdot)$ and refer to $K_A(\cdot,\cdot)$ simply as *regular conditional distribution of* A or as *Markov kernel of* A. Note that for every $A \in \mathcal{C}$, its conditional regular distribution $K_A(\cdot,\cdot)$, and every Borel set $G \in \mathcal{B}([0,1]^2)$ we have the following *disintegration* (here $G_x := \{y \in [0,1] : (x,y) \in G\}$ denotes the x-section of G for every $x \in [0,1]$)

$$\int_{[0,1]} K_A(x, G_x)\, d\lambda(x) = \mu_A(G), \tag{4}$$

so in particular

$$\int_{[0,1]} K_A(x, F)\, d\lambda(x) = \lambda(F) \tag{5}$$

for every $F \in \mathcal{B}([0,1])$. On the other hand, every Markov kernel $K : [0,1] \times \mathcal{B}([0,1]) \to [0,1]$ fulfilling (5) induces a unique element $\mu \in \mathcal{P}_{\mathcal{C}}([0,1]^2)$ via (4). For every $A \in \mathcal{C}$ and $x \in [0,1]$ the function $y \mapsto F_x^A(y) := K_A(x, [0,y])$ will be called *conditional distribution function of A at x*. For more details and properties of conditional expectation, regular conditional distributions, and disintegration see [9, 10]. For examples underlining the usefulness of Markov kernels in the copula setting we refer, for instance, to [19, 20]. For a general study of the interrelation between 2-increasingness and differential properties of copulas we refer to [7].

As direct application of the results in [12] the Markov kernel K_A of an arbitrary copula $A \in \mathcal{C}$ can be decomposed into the sum of three substochastic kernels K_A^a, K_A^s, K_A^d (from $[0,1]$ to $\mathcal{B}([0,1])$), i.e.

$$K_A(x, E) = K_A^a(x, E) + K_A^s(x, E) + K_A^d(x, E) \tag{6}$$

for every $x \in [0,1]$ and $E \in \mathcal{B}([0,1])$. Thereby, the measure $K_A^a(x, \cdot)$ is absolutely continuous with respect to λ, the measure $K_A^s(x, \cdot)$ is singular with respect to λ and has no point masses, and $K_A^d(x, \cdot)$ is discrete for every $x \in [0,1]$. Letting k_A denote the Radon-Nikodym derivative of μ_A with respect to λ_2 (almost everywhere) uniqueness of the kernel K_A implies that the measures $K_A^a(x, \cdot)$ and $E \mapsto \int_E k_A(x,y) d\lambda(y)$ coincide for almost all $x \in [0,1]$. In the sequel we will refer to the corresponding induced measures $\mu_A^a, \mu_A^s, \mu_A^d$, given by

$$\begin{aligned} \mu_A^a(E \times F) &= \int_E K_A^a(x, F) d\lambda(x), \qquad \mu_A^s(E \times F) = \int_E K_A^s(x, F) d\lambda(x) \\ \mu_A^d(E \times F) &= \int_E K_A^d(x, F) d\lambda(x) \end{aligned} \tag{7}$$

simply as absolutely continuous, discrete and singular components of μ_A. Letting A denote a (non-trivial) convex combination of the product copula

Π, the minimum copula M and a singular copula S whose conditional distribution functions are strictly increasing, continuous and have derivative zero a.e. (for a construction see [20]) it is straightforward to see that all three components $\mu_A^a, \mu_A^s, \mu_A^d$ are non-degenerated. We conclude this section with the following auxiliary result that we will use in Section 4:

Lemma 1. *For $A \in \mathcal{C}$ the following two conditions are equivalent:*

1. *A is singular*
2. *There exists a Borel set $\Lambda \subseteq [0,1]$ with $\lambda(\Lambda) = 1$ such that the measure $K_A(x, \cdot)$ is singular with respect to λ for every $x \in \Lambda$.*

Proof: If μ_A is singular, then, by definition, there exists a Borel set $N \subseteq [0,1]^2$ such that $\lambda_2(N) = 0$ and $\mu_A(N) = 1$. Applying disintegration to λ_2 and μ_A directly yields $\lambda(N_x) = 0$ and $K_A(x, N_x) = 1$ for almost every $x \in [0,1]$, which completes the proof of the first implication.
If the second condition holds then eq. (6) implies $K_A^a(x, [0,1]) = 0$ for almost every $x \in [0,1]$, from which we get $\int_{[0,1]^2} k_A(x,y) d\lambda_2(x,y) = 0$, i.e. the absolutely continuous component is degenerated and μ_A is singular. ■

3. Markov kernels of Archimedean copulas

For every generator $\varphi : [0,1] \to [0,\infty]$ we will let $D^+\varphi(x)$ ($D^-\varphi(x)$) denote the right-hand (left-hand) derivative of φ at $x \in (0,1)$. Convexity of φ implies that $D^+\varphi(x) = D^-\varphi(x)$ holds for all but at most countably many $x \in (0,1)$, i.e. φ is differentiable outside a countable subset of $(0,1)$, and that $D^+\varphi$ is non decreasing and right-continuous (see, for instance, [11, 15]). Setting $D^+\varphi(0) = -\infty$ in case of strict φ as well as $D^+\varphi(1) = 0$ (for strict and non-strict ones) allows to view $D^+\varphi$ as non-decreasing and right-continuous function on the full unit interval $[0,1]$. Additionally (again see [11, 15]) we have $D^-\varphi(x) = D^+\varphi(x-)$ for every $x \in (0,1)$.

If φ is strict define $K_\varphi(x, [0,y])$ for arbitrary $x, y \in [0,1]$ by (for every $a \in \mathbb{R}$ expressions of the from $\frac{a}{-\infty}$ are zero by definition throughout the whole paper)

$$K_\varphi(x, [0,y]) = \begin{cases} 1 & \text{if } x \in \{0,1\} \\ \frac{D^+\varphi(x)}{(D^+\varphi)(A_\varphi(x,y))} & \text{if } x \in (0,1). \end{cases} \tag{8}$$

If φ is non-strict let $K_\varphi(x,[0,y])$ be defined by

$$K_\varphi(x,[0,y]) = \begin{cases} 1 & \text{if } x \in \{0,1\} \\ \frac{D^+\varphi(x)}{(D^+\varphi)(A_\varphi(x,y))} & \text{if } x \in (0,1) \text{ and } y \geq f^0(x) \\ 0 & \text{if } x \in (0,1) \text{ and } y < f^0(x). \end{cases} \tag{9}$$

The following useful theorem holds:

Theorem 2. *If φ is strict then K_φ according to equation (8) defines a Markov kernel of A_φ. If φ is non-strict a Markov kernel of A_φ is given by (9).*

Proof: We only prove the result for strict φ - the case of non-strict φ can be proved analogously. Obviously $y \mapsto K_\varphi(x,[0,y])$ is a distribution function for $x \in \{0,1\}$. For $x \in (0,1)$ and $y \in \{0,1\}$ we obviously have $K_\varphi(x,[0,y]) = y$. Using the fact that $D^+\varphi$ is right-continuous and non-decreasing on $(0,1)$, it follows that $y \mapsto K_\varphi(x,[0,y])$ is a distribution function for $x \in (0,1)$ too. Extending $K_\varphi(x,\cdot)$ from the semiring $\{[0,y] : y \in [0,1]\}$ to $\mathcal{B}([0,1])$ therefore yields a probability measure $K_\varphi(x,\cdot)$ for every $x \in [0,1]$. On the other hand, for every fixed $y \in [0,1]$, the function $x \mapsto K_\varphi(x,[0,y])$ is measurable from which (using a standard Dynkin System argument) we get that $x \mapsto K_\varphi(x,B)$ is measurable for every Borel set $B \in \mathcal{B}([0,1])$. Altogether this implies that $K_\varphi(\cdot,\cdot)$ is a Markov kernel from $[0,1]$ to $\mathcal{B}([0,1])$ and it remains to show that $K_\varphi(\cdot,\cdot)$ is a Markov kernel of A_φ. Fix $y \in [0,1]$. Then, using convexity of φ^{-1} and bijectivity of φ, it follows that the set Λ of all points $x \in [0,1]$ at which $x \mapsto \varphi^{-1}(\varphi(x) + \varphi(y))$ is non-differentiable is at most countably infinite. Hence, using the chain rule we directly get $\int_{[0,x]} K_\varphi(t,[0,y])d\lambda(t) = A_\varphi(x,y)$ for every $x \in [0,1]$, from which the desired result follows immediately. ■.

The following two corollaries are well-known (see [14]) - the Markov kernel approach, however, allows for simplified and elegant alternative proofs. To simplify notation, let $E_{s,t} \subseteq [0,1]^2$ be defined by

$$E_{s,t} = \left\{(x,y) \in [0,1]^2 : x \leq s, A_\varphi(x,y) \leq t\right\} \tag{10}$$

for all $s,t \in [0,1]$.

Corollary 3. *Suppose that $s,t \in [0,1]$. If φ is a strict generator then we have $\mu_{A_\varphi}(E_{s,t}) = 0$ for $t = 0$ (and arbitrary s) as well as*

$$\mu_{A_\varphi}(E_{s,t}) = \begin{cases} s & \text{if } s \leq t \\ t + \frac{\varphi(s)-\varphi(t)}{D^+\varphi(t)} & \text{if } s > t. \end{cases} \tag{11}$$

for $t > 0$. If φ is non-strict then equation (11) holds for all $s, t \in [0,1]$.
As direct consequence, for arbitrary generator φ, the Kendall distribution function $F_{A_\varphi}^{Kendall}$ is given by

$$F_{A_\varphi}^{Kendall}(t) = t - \frac{\varphi(t)}{D^+\varphi(t)} \tag{12}$$

for every $t \in (0,1]$.

Proof: Since equation (12) directly follows from equation (11) by considering $s = 1$ it suffices to prove the first assertion.
Suppose that φ is strict. Because of $E_{s,0} \subseteq (\{0\} \times [0,1]) \cup ([0,1] \times \{0\})$ we directly get $\mu_{A_\varphi}(E_{s,0}) = 0$. For the case $t > 0$ we distinguish two cases: (i) If $s \leq t$ then, considering that $A_\varphi(x,y) \leq t$ is equivalent to $\varphi(x) + \varphi(y) \geq \varphi(t)$ and that $x \leq s$ implies $\varphi(x) \geq \varphi(s) \geq \varphi(t)$ the desired result follows immediately.
(ii) If $s > t$ then, using equality (4) we directly get

$$\begin{aligned}
\mu_{A_\varphi}(E_{s,t}) &= t + \int_{[t,s]} K_\varphi(x, [0, f^t(x)])d\lambda(x) \\
&= t + \int_{[t,s]} \frac{D^+\varphi(x)}{D^+\varphi\big(A_\varphi(x, f^t(x))\big)} d\lambda(x) = t + \int_{[t,s]} \frac{D^+\varphi(x)}{D^+\varphi(t)} d\lambda(x) \\
&= t + \frac{\varphi(s) - \varphi(t)}{D^+\varphi(t)},
\end{aligned}$$

which completes the proof for the case of strict φ.
In the case of non-strict φ there is no need to consider $t = 0$ and $t > 0$ separately and we can proceed completely analogous as in (i) and (ii) to get the desired result. ■

Corollary 4. *Suppose that φ is a generator. Then we have*

$$\mu_{A_\varphi}(L_t) = -\frac{\varphi(t)}{D^+\varphi(t)} + \frac{\varphi(t)}{D^+\varphi(t-)} = -\frac{\varphi(t)}{D^+\varphi(t)} + \frac{\varphi(t)}{D^-\varphi(t)} \tag{13}$$

for $t \in (0,1)$. Additionally, if φ is strict then $\mu_{A_\varphi}(L_0) = 0$ and if φ is non-strict then $\mu_{A_\varphi}(L_0) = -\frac{\varphi(0)}{D^+\varphi(0)}$ holds.

Proof 1: Set $E_r := \{(x,y) \in [0,1]^2 : A_\varphi(x,y) \leq r\}$ for every $r \in [0,1]$ and fix $t \in (0,1)$. Then, considering $L_t = E_t \setminus \bigcup_{n>1/t} E_{t-1/n}$ and using Corollary

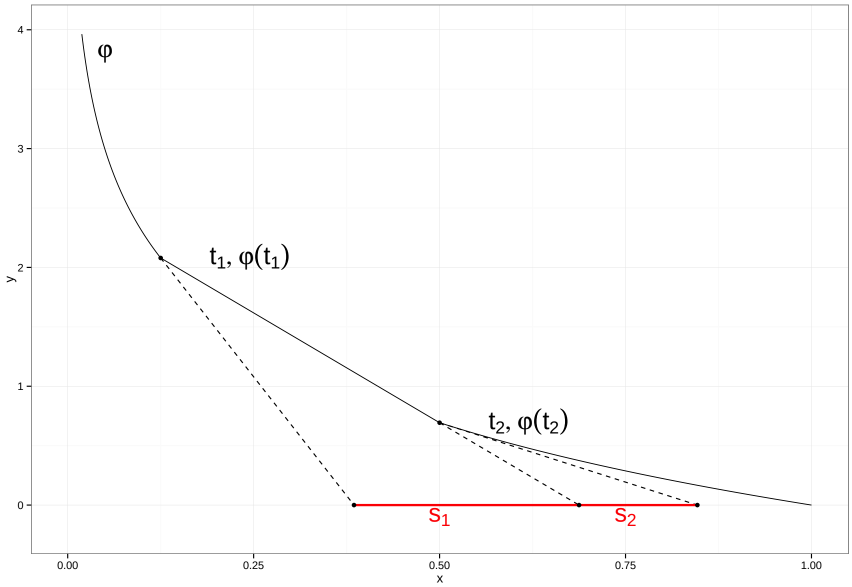

Figure 1: A strict generator φ for which $D^+\varphi$ is discontinuous at $t_1 = 1/8$ and $t_2 = 1/2$. The red segments s_1, s_2 have length $s_i = \varphi(t_i)\big(-\frac{1}{D^+\varphi(t_i)} + \frac{1}{D^-\varphi(t_i)}\big)$ for $i \in \{1, 2\}$. Figure 2 depicts a sample of the corresponding Archimedean copula A_φ, a histogram as well as the two corresponding marginal histograms.

3 we immediately get

$$\begin{aligned}
\mu_{A_\varphi}(L_t) &= \mu_{A_\varphi}(E_t) - \lim_{n\to\infty} \mu_{A_\varphi}(E_{t-1/n}) \\
&= t - \frac{\varphi(t)}{D^+\varphi(t)} - \lim_{n\to\infty}\left(t - 1/n - \frac{\varphi(t-1/n)}{D^+\varphi(t-1/n)}\right) \\
&= -\frac{\varphi(t)}{D^+\varphi(t)} + \frac{\varphi(t)}{D^+\varphi(t-)} = -\frac{\varphi(t)}{D^+\varphi(t)} + \frac{\varphi(t)}{D^-\varphi(t)}.
\end{aligned}$$

For non-strict φ the case $t = 0$ follows in the same way by using $L_0 = \bigcap_{n=1} E_{1/n}$, and $\mu_{A_\varphi}(L_0) = 0$ for strict φ is clear. ■

Proof 2: Consider $t \in (0, 1)$. Then, using equation (4), we get

$$\begin{aligned}
\mu_{A_\varphi}(L_t) &= \int_{[t,1]} K_\varphi(x, \{f^t(x)\}) d\lambda(x) = \int_{[t,1]} \frac{D^+\varphi(x)}{D^+\varphi(t)} - \frac{D^+\varphi(x)}{D^+\varphi(t-)} d\lambda(x) \\
&= -\frac{\varphi(t)}{D^+\varphi(t)} + \frac{\varphi(t)}{D^+\varphi(t-)} = -\frac{\varphi(t)}{D^+\varphi(t)} + \frac{\varphi(t)}{D^-\varphi(t)}.
\end{aligned}$$

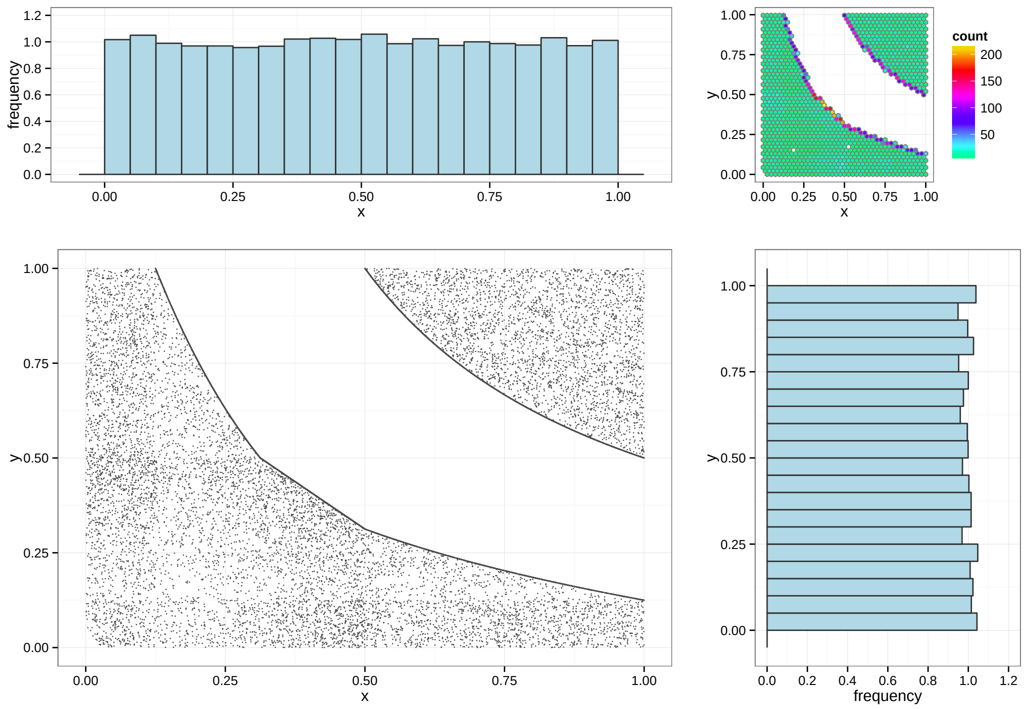

Figure 2: Sample of size 20.000 of the Archimedean copula A_φ with strict generator φ as depicted in Figure 1, its histogram as well as the two marginal histograms. The sample has been generated using Algorithm 3.5 in [1].

For non-strict φ and $t = 0$ we can use $\mu_{A_\varphi}(L_0) = \int_{[0,1]} K_\varphi(x, [0, f^0(x)])d\lambda(x)$ to get $\mu_{A_\varphi}(L_0) = -\frac{\varphi(0)}{D^+\varphi(0)}$. ■

Remark 5. Considering $K_\varphi(x, \{f^t(x)\})$ with f^t as before also shows how μ_{A_φ} distributes its mass (if any) on L_t. In particular, the function $x \mapsto K_\varphi(x, \{f^t(x)\})$ is non-increasing on $[t, 1]$.

Remark 6. Suppose that φ is strict and let $\mathcal{J}(D^+\varphi) \subseteq (0, 1)$ denote the set of all discontinuities of $D^+\varphi$. If $\mathcal{J}(D^+\varphi)$ is empty $\mu^d_{A_\varphi}([0, 1]) = 0$ follows, i.e. the discrete component of μ_{A_φ} is degenerated. In case of $\mathcal{J}(D^+\varphi) \neq \emptyset$ on the other hand we get

$$\mu^d_A([0,1]^2) = \sum_{t \in \mathcal{J}(D^+\varphi)} \varphi(t)\Big(-\frac{1}{D^+\varphi(t)} + \frac{1}{D^-\varphi(t)}\Big). \tag{14}$$

The latter sum also has a nice geometric interpretation as depicted in Figure 1 (also see [14]) - it coincides with the length of all line segments on the x-axis generated by left-and right-hand tangents at discontinuity points of $D^+\varphi$.

4. Singular Archimedean copulas with full support

Using the results from the previous section we can now prove the existence of singular Archimedean copulas with full support. If φ is non-strict then A_φ can not have full support (the interior of L_0 is non-empty and has no mass), hence we focus on strict generators.
As first result we construct an Archimedean copula A_φ whose conditional distribution functions $y \mapsto F_x^{A_\varphi}(y)$ are discrete and strictly increasing. In what follows $\beta_1 : \mathbb{N} \to \mathbb{Q} \cap [\frac{1}{2}, 1)$ denotes an arbitrary bijection. Given β_1, setting $\beta_i(n) := \frac{\beta_1(n)}{2^{i-1}}$, defines a bijection β_i from $\mathbb{N}$ onto $[\frac{1}{2^i}, \frac{1}{2^{i-1}}) \cap \mathbb{Q}$ for every $i \geq 2$. Choose an arbitrary sequence $(a_n)_{n\in\mathbb{N}}$ in $(-\infty, 0)$ such that $\sum_{n=1}^{\infty} |a_n| < \infty$ and define functions $F, F_1, F_2, \dots$ and φ on $[0,1]$ by

$$F_i(x) := \sum_{n=1}^{\infty} a_n \mathbf{1}_{[0,\beta_i(n)]}(x) \tag{15}$$

$$F(x) := \sum_{i=1}^{\infty} 2^i F_i(x) \tag{16}$$

$$\varphi(x) = \int_{[x,1]} -F d\lambda. \tag{17}$$

In the sequel $\mathcal{DC}(f)$ will denote the set of all discontinuity points of a function $f : [0,1] \to [-\infty, \infty]$.

Lemma 7. *F_i is a non-decreasing, left-continuous function with $\mathcal{DC}(F_i) = [\frac{1}{2^i}, \frac{1}{2^{i-1}}) \cap \mathbb{Q}$ and $\sum_{q\in\mathcal{DC}(F_i)}(F_i(q+) - F_i(q)) = -\sum_{i=1}^{\infty} a_i = F_i(1) - F_i(0)$, i.e. F_i is a non-decreasing, left-continuous jump function.*

Proof: Obviously F_i is non-decreasing and we have $F_i(\beta_i(n)+) - F_i(\beta_i(n)) \geq |a_n| > 0$ for every $n \in \mathbb{N}$. Considering $F_i(1) - F_i(0) = -\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} |a_i|$ we immediately get $\mathcal{DC}(F_i) = [\frac{1}{2^i}, \frac{1}{2^{i-1}}) \cap \mathbb{Q}$ and it remains to show that F_i is left-continuous on $[\frac{1}{2^i}, \frac{1}{2^{i-1}}) \cap \mathbb{Q}$. Since F_i is constant on $[0, \frac{1}{2^i}]$ it suffices to consider $q \in (\frac{1}{2^i}, \frac{1}{2^{i-1}}) \cap \mathbb{Q}$. Let $\varepsilon > 0$. By construction there exists a unique $n^\star \in \mathbb{N}$ with $\beta_i(n^\star) = q$ as well as some $n_0 \geq n^\star$ such that $\sum_{n=n_0}^{\infty} |a_n| < \varepsilon$.

Set $L := \{\beta_i(n) : n < n_0 \text{ and } \beta_i(n) < q\}$ and define $q_0 := \max(L)$ if $L \neq \emptyset$ and $q_0 := \frac{1}{2^i}$ otherwise. Then $\delta := q - q_0 > 0$ and for all $x \in (q - \delta, q)$ we obviously have $|F_i(q) - F_i(x)| < \varepsilon$, which completes the proof. ■

The next lemma summarizes the most important properties of F and φ.

Lemma 8. *F is a left-continuous, strictly increasing jump function with $F(0) = -\infty$ and $\mathcal{DC}(F) = [0, 1) \cap \mathbb{Q}$. Furthermore φ is a strict generator fulfilling $\mathcal{DC}(D^+\varphi) = \mathcal{DC}(F)$ and $D^+\varphi$ is a jump function.*

Proof: The facts that $F(0) = -\infty$ and that F is left-continuous follow directly from the construction. Furthermore, considering $\mathcal{DC}(F_i) = [\frac{1}{2^i}, \frac{1}{2^{i-1}}) \cap \mathbb{Q}$ together with equation (15) we get $\mathcal{DC}(F) = [0, 1) \cap \mathbb{Q}$, implying that F is strictly increasing. Additionally, F is a jump function since, by construction and Lemma 7, we have

$$F(1) - F(x) = \sum_{q \in [0,1) \cap \mathbb{Q} : q \geq x} (F(q+) - F(q)) \tag{18}$$

for every $x \in (0, 1]$. Since for every fixed $m \in \mathbb{N}$ we have

$$\begin{aligned} \varphi(0) &> \int_{[0,1]} \sum_{i=1}^{m} -2^i F_i \, d\lambda = \sum_{i=1}^{m} 2^i \int_{[0,1]} -F_i \, d\lambda \geq \sum_{i=1}^{m} \Big(2^i \frac{1}{2^i} \sum_{n=1}^{\infty} |a_n| \Big) \\ &= m \sum_{n=1}^{\infty} |a_n| \end{aligned}$$

Lebesgue's monotone convergence theorem (see [16]) implies $\varphi(0) = \infty$. Since $-F(x) > 0$ for every $x \in [0, 1)$ the function φ is strictly decreasing and it suffices to show that φ is convex. Convexity, however, is a direct consequence of eq. (17) and the fact that the antiderivative of a strictly increasing function is convex (see [15]).
In every continuity point $x \in (0, 1)$ of F we have $D^+\varphi(x) = F(x)$, so the functions $D^+\varphi$ and F coincide on $\mathbb{Q}^c \cap (0, 1)$, from which we get

$$\begin{aligned} F(x) &= F(x-) = D^+\varphi(x-) = D^-\varphi(x) \\ F(x+) &= D^+\varphi(x) \end{aligned}$$

for every $x \in \mathbb{Q} \cap (0, 1)$. Hence $\mathcal{DC}(D^+\varphi) = \mathcal{DC}(F)$ and $D^+\varphi$ and F even have the same jump heights. As direct consequence $D^+\varphi$ is a jump function too, which completes the proof. ■

With the help of the last two lemmas we arrive at the following result:

Theorem 9. *Let the strict generator φ be defined according to equation (17). Then the copula A_φ has the following properties:*

1. *A_φ is singular, has full support and we have $\mu^d_{A_\varphi}([0,1]^2) = 1$.*
2. *(Almost) all conditional distributions $F_x^{A_\varphi}$ are discrete with full support $[0,1]$.*
3. *The level curves L_t of A_φ fulfill: $\mu_{A_\varphi}(L_t) > 0$ if and only if $t \in \mathbb{Q} \cap (0,1)$.*
4. *$F^{Kendall}_{A_\varphi}$ is a discrete distribution function with full support $[0,1]$.*

Proof: We start with the second assertion and consider $x \in (0,1)$. The function $h_x : [0,1] \to [0,x]$, defined by $h_x(y) = A_\varphi(x,y)$ is a strictly increasing continuous bijection, hence $Q := h_x^{-1}(\mathbb{Q} \cap (0,1))$ is dense in $[0,1]$ and, using Lemma 8, $D^+\varphi \circ h_x$ is a strictly increasing, right-continuous jump function. As direct consequence, the function $G : [0,1] \to [0,1]$, defined by $G(y) = \frac{D^+\varphi(x)}{D^+\varphi \circ h_x(y)}$ is a strictly increasing distribution function, which has a discontinuity at each $y \in Q$ and which fulfills $G' = 0$ almost everywhere. To show that G is fully discrete let G^d denote the discrete component of the Lebesgue decomposition (see [5]) of G and suppose that $G^d(1) < 1$. Then the function $G^s(y) := \frac{D^+\varphi(x)}{D^+\varphi \circ h_x(y)} - G^d(y)$ is singular and it follows from the construction that so is the function

$$y \mapsto \frac{1}{G^s(y)} = \frac{D^+\varphi \circ h_x(y)}{D^+\varphi(x) - D^+\varphi \circ h_x(y) G^d(y)},$$

implying that the latter has no discontinuities, which is impossible. Hence we have $G^d(1) = 1$ and G is a discrete distribution function with full support, from which, taking into account eq. (8) and Theorem 2, the same follows for $F_x^{A_\varphi}$. As direct consequence, using disintegration (4) it follows that $\mu_{A_\varphi}(R) > 0$ holds for every rectangle $R \subseteq [0,1]^2$ with $\lambda_2(R) > 0$, implying that A_φ has full support $[0,1]^2$.

Using the fact that a copula is singular if and only if almost all conditional distributions are singular (Lemma 1) it follows immediately that A_φ is singular and that $\mu^d_{A_\varphi}([0,1]^2) = 1$, which completes the proof of the first assertion and, additionally, implies that

$$\sum_{t \in \mathbb{Q} \cap (0,1)} \varphi(t)\Big(-\frac{1}{D^+\varphi(t)} + \frac{1}{D^-\varphi(t)}\Big) = 1. \tag{19}$$

The third assertion is a direct consequence of Corrollary 4 and the construction of φ.

Finally, using eq. (12) we deduce that $F^{Kendall}_{A_\varphi}$ has a jump with height $\varphi(t)(-\frac{1}{D^+\varphi(t)} + \frac{1}{D^-\varphi(t)}) > 0$ at $t \in \mathbb{Q} \cap (0,1)$. Since, according to (19), all these heights sum up to one $F^{Kendall}_{A_\varphi}$ has to be discrete. ■

We now construct strict Archimedean copulas having the same properties as the copulas considered in [20], i.e. copulas A for which (almost all) conditional distribution functions F^A_x are singular (continuous and $(F^A_x)' = 0$ a.e.) and strictly increasing.

To do so, suppose that $g : [0,1] \to [-1,0]$ is continuous, strictly increasing and fulfills $g' = 0$ a.e. as well as $g(0) = -1$ and $g(1) = 0$ (for a construction of such functions see, for instance, [6, 8]). Given g, for every $i \in \mathbb{N}$ define a function $G_i : [0,1] \to [-1,0]$ by

$$G_i(x) = \begin{cases} -1 & \text{if } x \in \big[0, \frac{1}{2^i}\big) \\ g\big(\frac{x-1/2^i}{1/2^i}\big) & \text{if } x \in \big[\frac{1}{2^i}, \frac{1}{2^{i-1}}\big] \\ 0 & \text{if } x \in \big(\frac{1}{2^{i-1}}, 1\big] \end{cases} \tag{20}$$

and set

$$G(x) := \sum_{i=1}^{\infty} 2^i \, G_i(x) \tag{21}$$

$$\psi(x) := \int_{[x,1]} -G \, d\lambda \tag{22}$$

for every $x \in [0,1]$. The subsequent lemma gathers the most important properties of G and ψ.

Lemma 10. *G is strictly increasing and continuous on $(0,1]$ and fulfills $G' = 0$ a.e. Moreover ψ is a strict generator with $\psi'(x) = G(x)$ for every $x \in (0,1)$.*

Proof: It follows immediately from the construction that G is continuous on $(0,1]$, that G is strictly increasing on $(0,1]$ with $G(0) = -\infty, G(1) = 0$ and that $G' = 0$ a.e. To show $\psi(0) = \infty$ we may proceed analogously to the proof of Lemma 8, the fact that $\psi(1) = 0$ is clear by definition. Convexity is a direct consequence of eq. (22) and the afore-mentioned fact that the antiderivative of a strictly increasing function is convex (see [15]). The remaining assertion

$\psi'(x) = G(x)$ for every $x \in (0,1)$ follows from the fact that G is continuous on $(0,1]$. ■

Theorem 11. *Let the strict generator ψ be defined according to equation (22). Then the copula A_ψ has the following properties:*

1. *A_ψ is singular, has full support and we have $\mu^s_{A_\psi}([0,1]^2) = 1$.*
2. *(Almost) all conditional distribution functions $F_x^{A_\psi}$ are continuous, strictly increasing and singular.*
3. *Every level curve L_t of A_ψ fulfills $\mu_{A_\psi}(L_t) = 0$.*
4. *$F_{A_\psi}^{Kendall}$ is a strictly increasing singular distribution function.*

Proof: We again start with the proof of the second assertion and consider $x \in (0,1)$. The function $y \mapsto D^+\psi(A_\psi(x,y)) = G(A_\psi(x,y))$ is as composition of two strictly increasing continuous functions itself strictly increasing and continuous on $(0,1)$, from which, considering that ψ is a strict generator and using eq. (8) and Theorem 2 we get that $y \mapsto F_x^{A_\psi}(y)$ is a strictly increasing continuous function. Moreover, considering that the derivative h'_x of the bijection $h_x(y) := A_\psi(x,y)$ $(y \in [0,1])$ is positive and bounded away from zero on any interval $[a,b] \subseteq (0,1)$ according to [11] h_x can not map a set of strictly positive Lebesgue measure in a set of zero measure. Hence, letting $\Lambda \in \mathcal{B}([0,1])$ denote a set with $\lambda(\Lambda) = 1$ such that $G'(y) = 0$ for every $y \in \Lambda$, it follows that $\lambda(h_x^{-1}(\Lambda)) = 1$, implying the existence of a set $\Omega \in \mathcal{B}([0,1])$ such that $\lambda(\Omega) = 1$, h_x is differentiable at y and G is differentiable at $h_x(y)$ for every $y \in \Omega$. Having this, applying the chain rule and using $G' = 0$ a.e. directly yields $(F_x^{A_\psi})'(y) = 0$ a.e., which completes the proof of the second assertion.
The first assertion is a straightforward consequence of the second one, disintegration, and the characterization of singular copulas via their Markov kernels established in Lemma 1.
Since assertion three follows from Corollary 4 and the fact that G is continuous on $(0,1)$ it remains to prove assertion number four, which can be easily done as follows: Continuity of $\psi' = G$ on $(0,1)$ implies continuity of $F_{A_\psi}^{Kendall}$ on $[0,1]$ (left-continuity of $F_{A_\psi}^{Kendall}$ at 1 follows from the fact that $F_{A_\psi}^{Kendall}(t) \geq t$ for every $t \in (0,1)$). Moreover, letting $\Lambda \in \mathcal{B}((0,1))$ denote a set of full measure such that $G'(t) = 0$ for every $t \in \Lambda$ and using Corollary 3 we finally get

$$(F_{A_\psi}^{Kendall})'(t) = \frac{\psi(t)G'(t)}{(G(t))^2} = 0$$

for every $t \in \Lambda$. ■

As direct consequence of Theorem 11 we get the following result saying that Archimedean copulas can be smooth (differentiable with continuous derivative) and singular with full support at the same time.

Corollary 12. *There exist singular Archimedean copulas $A_\psi \in \mathcal{C}$ with full support fulfilling that $(x,y) \mapsto \frac{\partial}{\partial x} A_\psi(x,y)$ is continuous on $(0,1) \times [0,1]$ and $(x,y) \mapsto \frac{\partial}{\partial y} A_\psi(x,y)$ is continuous on $[0,1] \times (0,1)$.*

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