

A note on singularity of a recently introduced family of Minkowski's question-mark functions

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Abstract

We point out a mistake in the proof of the main theorem in a recent article on a family of generalized Minkowski's question-mark functions, saying that each member of the family is a singular homeomorphism, and provide two alternative proofs, one based on ergodicity of the Gauss map G and the α -Lüroth map L_α , and another one focusing more on classical properties of continued fraction expansions.

Résumé

Note sur la singularité d'une famille de fonctions "Minkowski's question-mark" récemment introduite. Nous mettons en évidence une erreur dans la démonstration du théorème principal dans un article récent traitant d'une famille de fonctions "Minkowski's question-mark" généralisées, stipulant que chaque membre de la famille est un homéomorphisme singulier, et nous produisons deux preuves alternatives, l'une basée sur l'ergodicité de l'application de Gauss G et de l'application α -Lüroth L_α , l'autre se focalisant plus sur des propriétés classiques des décompositions de fractions continues.

1. Introduction

Based on continued fraction- and generalized Lüroth expansions a new family of Minkowski question-mark functions was recently introduced in [1]. When proving the main theorem of the paper (Theorem 1.3) the author correctly shows that each member $?_\alpha$ of the family is a strictly increasing homeomorphism of the unit interval $[0, 1]$ and then tackles proving that $?_\alpha$ is singular (in the sense that $?_\alpha$ has derivative zero λ -a.e). Unfortunately, the presented proof of singularity is incorrect, implying that the theorem is still formally unproven. Main objective of this note is to provide two correct proofs of singularity of $?_\alpha$ for every partition α .

The rest of this note is organized as follows: We first introduce some notation (essentially following [1] with some slight modifications) in Section 2, point out what exactly went wrong in [1] and then prove Theorem 1.3 in [1] using two different methods in Section 3.

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2. Notation

In the sequel, $\mathcal{B}([0, 1])$ will denote the Borel σ -field on $[0, 1]$ and λ will denote the Lebesgue measure on $\mathcal{B}([0, 1])$. We will write $\mathbb{N}_\infty = \{1, 2, 3, \dots\} \cup \{\infty\}$ and will refer to $\Sigma := (\mathbb{N}_\infty)^\mathbb{N}$ as code-space. As usual, $G : [0, 1] \rightarrow [0, 1]$ will denote the Gauss map, defined by $G(0) = 0$ and $G(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$ for $x \in (0, 1]$. Set $s_i = \frac{1}{i}$ for every $i \in \mathbb{N}$ and $s_\infty = 0$. Then the intervals $I_\infty = \{s_\infty\}, I_1 = (s_2, s_1], I_2 = (s_3, s_2], \dots$ form a partition γ_G of $[0, 1]$. Coding orbits of G via γ_G , the continued fraction expansion $\text{cf} : [0, 1] \rightarrow \Sigma$ is defined by setting $\text{cf}(x) = \underline{a} = (a_1, a_2, a_3, \dots) \in \Sigma$ if and only if $G^{i-1}(x) \in I_{a_i}$ holds for every $i \in \mathbb{N}$. It is well known that G is strongly mixing (hence ergodic) w.r.t. the absolutely continuous probability measure μ_G with density $\frac{1}{\ln 2} \frac{1}{1+x}$ for $x \in [0, 1]$ (see [3]).

In the sequel we will let $\alpha = \{J_\infty, J_1, J_2, J_3, \dots\}$ denote partitions of the unit interval induced by strictly decreasing sequences $(t_i)_{i=1}^\infty$ converging to $0 =: t_\infty$ and fulfilling $t_1 = 1$, i.e. we have $J_\infty = \{t_\infty\}, J_1 = (t_2, t_1], J_2 = (t_3, t_2], \dots$. For each such partition α the α -Lüroth map L_α is defined by $L_\alpha(0) = 0$ as well as $L_\alpha(x) = \frac{t_j - x}{t_j - t_{j+1}}$ for $x \in (t_{j+1}, t_j]$ and $j \in \mathbb{N}$. Coding orbits of L_α via α , the α -Lüroth expansion $\text{Lür}_\alpha : [0, 1] \rightarrow \Sigma$ is defined by setting $\text{Lür}_\alpha(x) = \underline{a} = (a_1, a_2, a_3, \dots) \in \Sigma$ if and only if $L_\alpha^{i-1}(x) \in J_{a_i}$ holds for every $i \in \mathbb{N}$. Additionally to being strongly mixing w.r.t. λ , the α -Lüroth map is even (isomorphic to) a Bernoulli shift (see [2]). Moreover (again see [2]) the transformation $\Phi_\alpha : \Sigma \rightarrow [0, 1]$, defined by

$$\Phi_\alpha(\underline{a}) = t_{a_1} + \sum_{m=2}^{\infty} (-1)^{m+1} t_{a_m} \prod_{j=1}^{m-1} (t_{a_j} - t_{a_{j+1}}), \quad (1)$$

with the convention $t_{\infty+1} = t_\infty$, fulfills $\Phi_\alpha \circ \text{Lür}_\alpha = \text{id}_{[0,1]}$.

3. Two proofs of singularity of $?_\alpha$

Based on $\text{cf} : [0, 1] \rightarrow \Sigma$ and $\Phi_\alpha : \Sigma \rightarrow [0, 1]$, in [1] the author introduces the (generalized) question-mark function $?_\alpha$ by setting $?_\alpha(x) = \Phi_\alpha \circ \text{cf}(x)$ for every $x \in [0, 1]$ and then states the following theorem.

Theorem 3.1 ([1]) *Given a partition α as above, the map $?_\alpha : [0, 1] \rightarrow [0, 1]$ is an increasing singular homeomorphism fulfilling $L_\alpha \circ ?_\alpha = ?_\alpha \circ G$. Moreover, if $t_1, t_2, \dots \in \mathbb{Q}$, then $?_\alpha$ maps the set \mathbb{A}_2 of all quadratic surds into \mathbb{Q} .*

In [1] a correct proof for the fact that $?_\alpha$ is an increasing homeomorphism and for the assertion concerning \mathbb{A}_2 is given. It is well known that singularity can be shown by establishing the existence of a Borel set $\tilde{B} \subseteq [0, 1]$ fulfilling $\lambda(\tilde{B}) = 0$ and $\lambda(?_\alpha(\tilde{B})) = 1$. Letting μ_α denote the pull-back of λ via $?_\alpha$ (or, equivalently, the push-forward of λ via $?_\alpha^{-1}$), defined by $\mu_\alpha(B) = \lambda(?_\alpha(B))$ for every Borel set $B \in \mathcal{B}([0, 1])$, the author then uses the identity

$$\int_{?_\alpha(B)} dx = \int_B ?_\alpha^{-1}(x) dx \quad (2)$$

to prove that μ_α and μ_G are singular with respect to each other. Eq. (2), however, is easily seen to be wrong and should be $\int_{?_\alpha(B)} 1 dx = \int_B 1 d\mu_\alpha = \mu_\alpha(B)$ instead. In fact, considering, for instance, $B = [0, 1]$, we get $\int_{?_\alpha([0,1])} dx = \int_{[0,1]} dx = 1$, whereas the right-hand side of eq. (2) obviously fulfils $\int_{[0,1]} ?_\alpha^{-1}(x) dx < 1$ since $?_\alpha^{-1}$ is a homeomorphism of $[0, 1]$ too. Since the rest of the proof in [1] builds upon eq. (2), an alternative method is needed to show singularity of $?_\alpha$.

We will now provide two proofs of singularity of $\tilde{?}_\alpha$ for every partition α - the first one uses ergodicity of G and L_α and explicitly constructs a Borel set \tilde{B} with the afore-mentioned properties, the second one is more elementary and directly derives the fact $\tilde{?}'_\alpha(x) = 0$ for λ -a.e. $x \in [0, 1]$ via properties of continued fraction expansions.

Proof a: We distinguish two different types of partitions α :

Type I: There exists $k \in \mathbb{N}$ such that $t_k - t_{k+1} \neq \frac{1}{\ln 2} \ln \frac{(k+1)^2}{k(k+2)}$.

Let the Borel sets Λ and Γ be defined by

$$\Lambda = \left\{ x \in [0, 1] : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_{(\frac{1}{k+1}, \frac{1}{k}]} \circ G^i(x) = \frac{1}{\ln 2} \ln \frac{(k+1)^2}{k(k+2)} \text{ holds for every } k \in \mathbb{N} \right\} \quad (3)$$

$$\Gamma = \left\{ x \in [0, 1] : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_{(t_{k+1}, t_k]} \circ L_\alpha^i(x) = t_k - t_{k+1} \text{ holds for every } k \in \mathbb{N} \right\}. \quad (4)$$

Ergodicity of G w.r.t. μ_G and of L_α w.r.t. λ (see [2,3]) and the fact that μ_G is absolutely continuous with strictly positive density implies $\lambda(\Lambda) = \lambda(\Gamma) = 1$. Considering that $\tilde{?}_\alpha(x) = \Phi_\alpha \circ \text{cf}(x)$ holds for every $x \in (0, 1]$ and that for every $x \in \Lambda$ we have $\text{cf}(x) \notin \text{Lür}_\alpha(\Gamma)$, $\tilde{?}_\alpha(\Lambda) \subseteq \Gamma^c$ follows. Hence, choosing $\tilde{B} = \Lambda^c$ directly yields $\lambda(\tilde{B}) = 0$ as well as $1 \geq \lambda(\tilde{?}_\alpha(\tilde{B})) \geq \lambda(\Gamma) = 1$, which completes the proof for all partitions α of Type I.

Type II: For every $k \in \mathbb{N}$ we have $t_k - t_{k+1} = \frac{1}{\ln 2} \ln \frac{(k+1)^2}{k(k+2)}$.

Taking into account $t_1 = 1$ we get that there is only one partition α of Type II, namely the one fulfilling $t_k = \frac{1}{\ln 2} \ln \frac{k+2}{k+1}$ for every $k \in \mathbb{N}$. Instead of considering asymptotic frequencies of single 'digits' of the Lüroth- and continued-fraction expansions we now consider the asymptotic frequency of the 'block' $(1, 1)$ and set

$$\Lambda = \left\{ x \in [0, 1] : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_{(\frac{1}{2}, 1]^2} \circ (G^i(x), G^{i+1}(x)) = \frac{1}{\ln 2} \ln \frac{10}{9} \right\} \quad (5)$$

$$\Gamma = \left\{ x \in [0, 1] : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_{(\frac{1}{2}, 1]^2} \circ (L_\alpha^i(x), L_\alpha^{i+1}(x)) = \left(\frac{\ln \frac{4}{3}}{\ln 2} \right)^2 \right\}. \quad (6)$$

According to Proposition 4.1.2 in [3] we have $\lambda(\Lambda) = 1$. Moreover, using the fact that L_α is (isomorphic to) a Bernoulli shift and that $(t_1 - t_2)^2 = \left(\frac{\ln \frac{4}{3}}{\ln 2} \right)^2$ holds, $\lambda(\Gamma) = 1$ follows. Considering $\left(\frac{\ln \frac{4}{3}}{\ln 2} \right)^2 \neq \frac{\ln \frac{10}{9}}{\ln 2}$ and proceeding as in the second part of the first case and setting $\tilde{B} = \Lambda^c$ completes the proof. ■

Remark 1 Instead of using Proposition 4.1.2 in [3] and the fact that L_α is a Bernoulli shift in order to prove $\lambda(\Lambda) = \lambda(\Gamma) = 1$ for α of Type II, we could as well consider the maps $\varepsilon_G, \varepsilon_{L_\alpha} : [0, 1] \rightarrow [0, 1]^2$, defined by $\varepsilon_G(x) = (x, G(x))$ and $\varepsilon_{L_\alpha}(x) = (x, L_\alpha(x))$, and directly work with ergodicity of G and L_α . In fact, applying Birkhoff's ergodic theorem ([4]) to the indicator function $f : x \mapsto \mathbf{1}_{(\frac{1}{2}, 1]^2} \circ \varepsilon_G(x)$ directly yields that for μ_G -a.e. $x \in [0, 1]$ (hence for λ -a.e. $x \in [0, 1]$) we get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_{(\frac{1}{2}, 1]^2} \circ (G^i(x), G^{i+1}(x)) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} f \circ G^i(x) = \frac{1}{\ln 2} \int_{[0, 1]} f(x) \frac{1}{1+x} dx = \frac{1}{\ln 2} \ln \frac{10}{9}.$$

Proceeding analogously with the function $f : x \mapsto \mathbf{1}_{(\frac{1}{2}, 1]^2} \circ \varepsilon_{L_\alpha}(x)$ shows $\lambda(\Gamma) = 1$.

Proof b: Let Λ' denote the set of all points $x \in (0, 1)$ at which $?_\alpha$ is differentiable and define Λ according to eq. (3). For every $k \in \mathbb{N}$ define two new Borel sets Λ_1^k, Λ_3^k by

$$\Lambda_1^k = \left\{ x \in [0, 1] : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_{(\frac{1}{2}, 1] \times (\frac{1}{k+1}, \frac{1}{k}]} \circ (G^i(x), G^{i+1}(x)) = \frac{1}{\ln 2} \ln \frac{1 + \frac{k+1}{k+2}}{1 + \frac{k}{k+1}} \right\}$$

$$\Lambda_3^k = \left\{ x \in [0, 1] : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_{(\frac{1}{4}, \frac{1}{3}] \times (\frac{1}{k+1}, \frac{1}{k}]} \circ (G^i(x), G^{i+1}(x)) = \frac{1}{\ln 2} \ln \frac{1 + \frac{k+1}{3k+4}}{1 + \frac{k}{3k+1}} \right\}.$$

Following the same reasoning as in Remark 1 (or using Proposition 4.1.2 in [3]) we get $\lambda(\Lambda_1^k \cap \Lambda_3^k) = 1$, implying that $\Omega := \Lambda' \cap \Lambda \cap \bigcap_{k=1}^{\infty} (\Lambda_1^k \cap \Lambda_3^k)$ fulfills $\lambda(\Omega) = 1$. Fix $x \in \Omega$ and set $\underline{a} := cf(x) \in \Sigma$. Additionally, for every $n \geq 3$ let $x_n, y_n \in \mathbb{Q} \cap [0, 1]$ be defined by

$$x_n = [a_1, a_2, \dots, a_n] := \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots + \frac{1}{a_n}}}}, \quad y_n = [a_1, a_2, \dots, a_n + 1] := \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots + \frac{1}{a_n + 1}}}}. \quad (7)$$

Then we have $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = x$ as well as $x \in (x_n, y_n)$ for n even and $x \in (y_n, x_n)$ for n odd. Setting $x_n = \frac{p_n}{q_n}$ with p_n, q_n relatively prime and using the recurrence relations (1.12) in [2] we get $y_n = \frac{(a_n+1)p_{n-1} + p_{n-2}}{(a_n+1)q_{n-1} + q_{n-2}} = \frac{p_n + p_{n-1}}{q_n + q_{n-1}}$, which implies $x_n - y_n = (-1)^{n+1} \frac{1}{q_n(q_n + q_{n-1})}$. On the other hand, considering $?_\alpha(x_n) = \Phi_\alpha((a_1, a_2, \dots, a_n, \infty, \infty, \dots))$ and $?_\alpha(y_n) = \Phi_\alpha((a_1, a_2, \dots, a_n + 1, \infty, \infty, \dots))$, using eq. (1) yields

$$\delta_n := \frac{?_\alpha(x_n) - ?_\alpha(y_n)}{x_n - y_n} = \frac{(-1)^{n+1} \prod_{j=1}^n (t_{a_j} - t_{a_j+1})}{(-1)^{n+1} \frac{1}{q_n(q_n + q_{n-1})}} = q_n(q_n + q_{n-1}) \prod_{j=1}^n (t_{a_j} - t_{a_j+1}) \geq 0 \quad (8)$$

Since by construction $?_\alpha$ is differentiable at x , obviously $\lim_{n \rightarrow \infty} \delta_n = ?'_\alpha(x) \geq 0$ holds.

Suppose now that $?'_\alpha(x) > 0$. Then $\lim_{n \rightarrow \infty} \frac{\delta_n}{\delta_{n-1}} = 1$ follows, from which, again using the recurrence relations (1.12) in [2], we get

$$1 = \lim_{n \rightarrow \infty} \frac{\delta_n}{\delta_{n-1}} = \lim_{n \rightarrow \infty} (t_{a_n} - t_{a_n+1}) \frac{q_n(q_n + q_{n-1})}{q_{n-1}(q_{n-1} + q_{n-2})} = \lim_{n \rightarrow \infty} (t_{a_n} - t_{a_n+1}) \frac{\frac{q_n^2}{q_{n-1}} + \frac{q_n}{q_{n-1}}}{1 + \frac{q_{n-2}}{q_{n-1}}}$$

$$= \lim_{n \rightarrow \infty} (t_{a_n} - t_{a_n+1}) \frac{(a_n + \frac{q_{n-2}}{q_{n-1}})^2 + (a_n + \frac{q_{n-2}}{q_{n-1}})}{1 + \frac{q_{n-2}}{q_{n-1}}}. \quad (9)$$

Fix an arbitrary $k \in \mathbb{N}$. Letting let $(n_j)_{j \in \mathbb{N}}$ denote the subsequence of all indices with $a_{n_j} = k$, eq. (9) simplifies to

$$1 = (t_k - t_{k+1}) \lim_{j \rightarrow \infty} \frac{(k + \frac{q_{n_j-2}}{q_{n_j-1}})^2 + (k + \frac{q_{n_j-2}}{q_{n_j-1}})}{1 + \frac{q_{n_j-2}}{q_{n_j-1}}} \quad (10)$$

By construction of Ω we have that $(a_{n_j-1}, a_{n_j}) = (1, k)$ is fulfilled infinitely often and that the same is true for $(a_{n_j-1}, a_{n_j}) = (3, k)$. Using the same notation as in eq. (7), according to [2], $\frac{q_{n_j-2}}{q_{n_j-1}} = [a_{n_j-1}, \dots, a_2, a_1]$ holds, from which we conclude that $\frac{q_{n_j-2}}{q_{n_j-1}}$ lies infinitely often in $(\frac{1}{2}, 1]$ and infinitely often in $(\frac{1}{4}, \frac{1}{3}]$. This contradicts eq. (10), implying that we can not have $?'_\alpha(x) > 0$ if $x \in \Omega$. Since $x \in \Omega$ was arbitrary, we have shown that $?'_\alpha = 0$ holds λ -a.e., which completes the proof. ■

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