Maximal asymmetry of bivariate copulas and consequences to measures of dependence

Florian Griessenberger*, Wolfgang Trutschnig†

Abstract

In this paper we focus on copulas underlying maximal non-exchangeable pairs (X, Y) of continuous random variables X, Y either in the sense of the uniform metric d_∞ or the conditioning-based metrics D_p, and analyze their possible extent of dependence quantified by the recently introduced dependence measures ζ_1 and ξ. Considering maximal d_∞-asymmetry we obtain $\zeta_1 \in \left[\frac{5}{6}, 1\right]$ and $\xi \in \left[\frac{2}{3}, 1\right]$, in the case of maximal D_1-asymmetry we get $\zeta_1 \in \left[\frac{3}{4}, 1\right]$ and $\xi \in \left(\frac{1}{2}, 1\right]$, implying that maximal asymmetry implies a very high degree of dependence in both cases. Furthermore, we study various topological properties of the family of copulas with maximal D_1-asymmetry and derive some surprising properties for maximal D_p-asymmetric copulas.

Keywords: Asymmetry, copula, dependence measure, exchangeability, Markov kernel

1 Introduction

Two random variables X and Y with joint distribution function H are called exchangeable if and only if the pairs (X, Y) and (Y, X) have the same distribution, or equivalently, if $H(x, y) = H(y, x)$ holds for all x and y. The study of exchangeable random variables has exhibited a lot of interest in statistics (see, for instance, [8] and the references therein). In case X and Y are identically distributed and have distribution function F, then (X, Y) is exchangeable if and only if the underlying copula A coincides with its transpose A^t (defined as $A^t(x, y) = A(y, x)$). Hence, in what follows we consider continuous and identically distributed random variables X and Y. While the class of continuous exchangeable random variables X and Y is uniquely characterized by the class of symmetric copulas, the exact opposite, i.e. maximal non-exchangeability of random variables, strongly depends on the choice of measure quantifying the degree of non-exchangeability. One natural measure of non-exchangeability was studied by Nelsen [21] as well as by Klement and Mesiar [15], who independently showed that

$$d_\infty(A, A^t) := \sup_{x,y\in[0,1]} |A(x, y) - A(y, x)| \leq \frac{1}{3}$$

holds for every $A \in \mathcal{C}$ and introduced the d_∞-based measure $\delta : \mathcal{C} \to [0, 1]$ via $\delta(A) := 3d_\infty(A, A^t)$. Moreover, they characterized all copulas $A \in \mathcal{C}$ with maximal d_∞-asymmetry and showed that these copulas always model slightly negatively correlated random variables X and Y in the sense

*University of Salzburg, Department for Artificial Intelligence & Human Interfaces, Hellbrunnerstrasse 34, 5020 Salzburg, E-mail: florian@griessenberger.net

†University of Salzburg, Department for Artificial Intelligence & Human Interfaces, Hellbrunnerstrasse 34, 5020 Salzburg, wolfgang@trutschnig.net

of Spearman's ρ. More precisely, $\delta(A) = 1$ implies $\rho(A) \in \left[-\frac{5}{9}, -\frac{1}{3}\right]$. Similar results also hold for different measures of concordance (see [17]).

Considering other metrics on the space of copulas yields alternative measures of non-exchangeability ([13, 25]): In [13] the stronger conditioning-based metric D_1 introduced in [27] was studied and the authors proved (among other things) that every copula $A \in \mathcal{C}$ with maximal D_1-asymmetry (i.e., $D_1(A, A^t) = \frac{1}{2}$) is not maximal asymmetric with respect to d_∞ and that no maximal d_∞-asymmetric copula is maximal asymmetric with respect to D_1.

Building upon the results in [13] we here further investigate the family of copulas with maximal D_1-asymmetry, derive additional novel characterizations in terms of the Markov-product of copulas (see [3]), and study various topological properties; inter alia we prove that the family of mutually completely dependent copulas with maximal D_1-asymmetry is dense in the set of all copulas with maximal D_1-asymmetry. Furthermore, we extend the concept of maximal D_1-asymmetry to the general D_p-metrics ($p \in [1, \infty)$), defined by

$$D_p(A, B) := \left(\int_{[0,1]} \int_{[0,1]} |K_A(x, [0, y]) - K_B(x, [0, y])|^p \, d\lambda(x) d\lambda(y) \right)^{\frac{1}{p}}, \tag{1}$$

where $K_A(\cdot, \cdot), K_B(\cdot, \cdot)$ denote the Markov kernels (regular conditional distributions) of $A, B \in \mathcal{C}$, respectively. Although all D_p-metrics induce the same topology, we show the surprising result that maximal D_1-asymmetry is not equivalent to maximal D_p-asymmetry for $p \in (1, \infty)$. In fact, copulas with maximal D_p-asymmetry with $p \in (1, \infty)$ are always mutually completely dependent and maximal asymmetric w.r.t. D_1.

Moreover, we tackle the question on the degree of dependence of copulas exhibiting maximal asymmetry with respect to d_∞ or D_p for every $p \in [1, \infty]$. Since measures of concordance are generally not suitable for quantifying dependence (see, for instance, [11]) we consider the dependence measures ζ_1 introduced in [27] and further studied in [11, 10], as well as ξ, defined in [4] and reinvestigated in [2]. Both measures have recently attracted a lot of interest (see, e.g., [1, 10, 11, 14, 24, 26]) since, in contrast to standard methods like Spearman's ρ or Kendall's τ, these measures are 1 if and only if Y is a function of X and 0 if and only if X and Y are independent; moreover, they can be estimated consistently without underlying smoothness assumptions. We prove that when considering maximal d_∞-asymmetry $\zeta_1 \in \left[\frac{5}{6}, 1\right]$ and $\xi \in \left[\frac{2}{3}, 1\right]$ hold, and in the case of maximal D_1-asymmetry $\zeta_1 \in \left[\frac{3}{4}, 1\right]$ and $\xi \in \left(\frac{1}{2}, 1\right]$ follows. In other words, maximal non-exchangeable random variables (in the sense of d_∞ or D_p) always implies a high degree of dependence w.r.t. ζ_1 and ξ.

The rest of this paper is organized as follows: Section 2 gathers preliminaries and notations that will be used throughout the paper. In Section 3 we study possible values of ζ_1 and ξ for maximal d_∞-asymmetric copulas and discuss an example illustrating differences of ζ_1 and ξ in the context of ordinal sums. In Section 4 we revisit copulas with maximal D_1-asymmetry and derive several topological properties. Extensions on maximal D_p-asymmetry for $p \in [1, \infty]$ and some interrelations are established in Section 5. Consequences on the dependence measures ζ_1 and ξ conclude the paper (Section 6). Various examples and graphics illustrate both the obtained results and the ideas underlying the proofs.

2 Notation & preliminaries

For every metric space (Ω, d) the Borel σ-field in Ω will be denoted by $\mathcal{B}(\Omega)$, λ will denote the Lebesgue measure on $\mathcal{B}(\mathbb{R})$. $\mathcal{T}$ will denote the class of all measurable λ-preserving transformations on $[0,1]$, i.e.,

$$\mathcal{T} = \{T\colon [0,1] \to [0,1] \text{ measurable with } \lambda(T^{-1}(E)) = \lambda(E) \quad \forall E \in \mathcal{B}([0,1])\},$$

and $\mathcal{T}_b$ the subclass of all bijective $T \in \mathcal{T}$. Throughout the whole paper $\mathcal{C}$ will denote the family of all two-dimensional copulas, $\mathcal{P}$ the family of all doubly-stochastic measures (for background on copulas and doubly stochastic measures we refer to [6, 22] and the references therein). Furthermore, M denotes the upper Fréchet Hoeffding bound, Π the product copula and W the lower Fréchet Hoeffding bound. Additionally, the completely dependent copula induced by a measure-preserving transformation $h \in \mathcal{T}$ will be denoted by C_h (see [27], Definition 9). The family of all completely dependent copulas will be denoted by $\mathcal{C}_{cd}$, the family of all mutually completely dependent copulas by $\mathcal{C}_{mcd} := \{C_h \in \mathcal{C}_{cd} : h \in \mathcal{T}_b\}$. For every copula $C \in \mathcal{C}$ the corresponding doubly stochastic measure will be denoted by μ_C. As usual, d_∞ denotes the uniform metric on $\mathcal{C}$, i.e.,

$$d_\infty(A,B) := \max_{(x,y)\in[0,1]^2} |A(x,y) - B(x,y)|$$

for every $A, B \in \mathcal{C}$. It is well known that $(\mathcal{C}, d_\infty)$ is a compact metric space (see [6]).

In what follows Markov kernels will play an important role. A mapping $K\colon \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0,1]$ is called a Markov kernel from $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ if the mapping $x \mapsto K(x,B)$ is measurable for every fixed $B \in \mathcal{B}(\mathbb{R})$ and the mapping $B \mapsto K(x,B)$ is a probability measure for every fixed $x \in \mathbb{R}$. A Markov kernel $K\colon \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0,1]$ is called regular conditional distribution of a (real-valued) random variable Y given (another random variable) X if for every $B \in \mathcal{B}(\mathbb{R})$

$$K(X(\omega), B) = \mathbb{E}(\mathbb{1}_B \circ Y | X)(\omega)$$

holds $\mathbb{P}$-a.s. It is well known that a regular conditional distribution of Y given X exists and is unique $\mathbb{P}^X$-almost sure (where $\mathbb{P}^X$ denotes the distribution of X, i.e., the push-forward of $\mathbb{P}$ via X). For every $A \in \mathcal{C}$ (a version of) the corresponding regular conditional distribution (i.e., the regular conditional distribution of Y given X in the case that $(X,Y) \sim A$) will be denoted by $K_A(\cdot,\cdot)$. Note that for every $A \in \mathcal{C}$ and Borel sets $E, F \in \mathcal{B}([0,1])$ we have

$$\int_E K_A(x,F)\, d\lambda(x) = \mu_A(E \times F) \quad \text{and} \quad \int_{[0,1]} K_A(x,F)\, d\lambda(x) = \lambda(F). \tag{2}$$

For more details and properties of conditional expectations and regular conditional distributions we refer to [12, 16]. Expressing copulas in terms of their corresponding regular conditional distribution yields metrics stronger than d_∞ (see [27]) and defined by

$$D_p(A,B) := \left(\int_{[0,1]} \int_{[0,1]} |K_A(x,[0,y]) - K_B(x,[0,y])|^p \, d\lambda(x) d\lambda(y) \right)^{\frac{1}{p}}, \tag{3}$$

$$D_\infty(A,B) := \sup_{y\in[0,1]} \int_{[0,1]} |K_A(x,[0,y]) - K_B(x,[0,y])| \, d\lambda(x). \tag{4}$$

To simplify notation we will also write $\Phi_{A,B}(y) := \int_{[0,1]} |K_A(x,[0,y]) - K_B(x,[0,y])| d\lambda(x)$. We will also work with D_∂, defined by

$$D_\partial(A,B) := D_1(A,B) + D_1(A^t, B^t),$$

whereby A^t denotes the transpose of $A \in \mathcal{C}$. The metric D_∂ can be seen as metrization of the so-called ∂-convergence, introduced and studied in [18, 19]. In [27] it is shown that $(\mathcal{C}, D_1)$ is a complete and separable metric space with diameter $1/2$ and that the topology induced by D_1 is strictly finer than the one induced by d_∞. For further background on D_1 and D_∂ as well as for possible extensions to the multivariate setting we refer to [6, 7, 10, 27] and the references therein. The D_1-based dependence measure ζ_1 (introduced in [27] and further investigated in [10, 11]) is defined as

$$\zeta_1(X, Y) := \zeta_1(A) := 3D_1(A, \Pi),$$

whereby (X, Y) has copula $A \in \mathcal{C}$. In the sequel we will also consider the dependence measure ξ (first introduced in [4] and reinvestigated in [2]) defined as

$$\xi(X, Y) := \frac{\int Var(\mathbb{E}(\mathbb{1}_{\{Y \geq t\}}|X))d\mu(t)}{\int Var(\mathbb{1}_{\{Y \geq t\}})d\mu(t)},$$

where μ is the law of Y. In the copula setting, it is straightforward to verify that ξ can be expressed in terms of D_2 and that $\xi(X, Y) := \xi(A) = 6D_2^2(A, \Pi)$ holds. Both dependence measures attain values in $[0, 1]$ and are 0 if and only if $A = \Pi$, and 1 if and only if A is completely dependent. Letting $S_h(A)$ denote the generalized shuffle of A w.r.t. the first coordinate, implicitly defined via the corresponding doubly stochastic measure μ_A by

$$\mu_{\mathcal{S}_h(A)}(E \times F) := \mu_A(h^{-1}(E) \times F),$$

for all $E, F \in \mathcal{B}([0, 1])$ (see, e.g., [5, 9]), the following simple result holds:

Lemma 2.1. *Let $h \in \mathcal{T}_b$ be a λ-preserving bijection. Then $\zeta_1(S_h(A)) = \zeta_1(A)$ and $\xi(S_h(A)) = \xi(A)$ holds for every $A \in \mathcal{C}$.*

Proof. According to Lemma 3.1 in [9] for $h \in \mathcal{T}_b$ the Markov kernel of $\mathcal{S}_h(A)$ can be expressed as $K_{\mathcal{S}_h(A)}(x, [0, y]) = K_A(h^{-1}(x), [0, y])$ and for $p \in [1, \infty)$ we get

$$\begin{aligned}
D_p^p(\mathcal{S}_h(A), \Pi) &= \int_{[0,1]} \int_{[0,1]} |K_{\mathcal{S}_h(A)}(x, [0, y]) - y|^p d\lambda(x) d\lambda(y) \\
&= \int_{[0,1]} \int_{[0,1]} |K_A(h^{-1}(x), [0, y]) - y|^p d\lambda(x) d\lambda(y) \\
&= \int_{[0,1]} \int_{[0,1]} |K_A(h^{-1}(x), [0, y]) - y|^p d\lambda^h(x) d\lambda(y) \\
&= \int_{[0,1]} \int_{[0,1]} |K_A(h^{-1}(h(x)), [0, y]) - y|^p d\lambda(x) d\lambda(y) = D_p^p(A, \Pi),
\end{aligned}$$

which proves the assertion. □

In the sequel we will also work with rearrangements [23] (see [26] for an elegant application of rearrangements in the copula context). We call $f^* : [0, 1] \to \mathbb{R}$ the decreasing rearrangement of a Borel measurable function $f : [0, 1] \to \mathbb{R}$ if it fulfills $f^*(t) := \inf\{x \in \mathbb{R} : \lambda(\{z \in [0, 1] : f(z) > x\}) \leq t\}$. The stochastically increasing (SI)-rearrangement $A^\uparrow$ of A is then defined as

$$A^\uparrow(x, y) := \int_{[0,x]} K_A(t, [0, y])^* \, d\lambda(t),$$

whereby the rearrangement is applied on the first coordinate of $K_A(\cdot, \cdot)$, i.e., for every fixed $y \in [0, 1]$ the rearranged Markov kernel is defined via $K_A(t, [0, y])^* := \inf\{x \in [0, 1] : \lambda(\{z \in [0, 1] :$

$K_A(z,[0,y]) > x\}) \le t\}$. In [26] it was shown that $A^{\uparrow}$ is a stochastically increasing copula and both dependence measures ζ_1 and ξ are invariant w.r.t. to the rearrangement, i.e., they fulfil $\zeta_1(A^{\uparrow}) = \zeta_1(A)$ and $\xi(A^{\uparrow}) = \xi(A)$, respectively. Recall that a copula A is called stochastically increasing (SI) if there exists a Borel set $\Lambda \subseteq [0,1]$ with $\lambda(\Lambda) = 1$ such that for any $y \in [0,1]$ the mapping $x \mapsto K_A(x,[0,y])$ is non-increasing on Λ. The family of all stochastically increasing copulas will be denoted by $\mathcal{C}^{\uparrow}$. For further information we refer to [22] and the references therein.

Given $A, B \in \mathcal{C}$ a new copula denoted by $A*B$ can be constructed via the so-called star/Markov product $A*B$ (see [3]) by

$$(A*B)(x,y) := \int_{[0,1]} \partial_2 A(x,t)\partial_1 B(t,y)\, d\lambda(t), \tag{5}$$

where $\partial_1 A(x,y)$ denotes the partial derivative of A with respect to the first coordinate. The star product $A*B$ is always a copula, i.e., no smoothness assumptions on A, B are required. Translating to the Markov kernel setting the star product corresponds to the well known composition of Markov kernels and the following lemma holds:

Lemma 2.2 ([29]). *Suppose that $A, B \in \mathcal{C}$ and let K_A, K_B denote the Markov kernels of A and B, respectively. Then the Markov kernel $K_A \circ K_B$, defined by*

$$(K_A \circ K_B)(x,F) := \int_{[0,1]} K_B(y,F)\, K_A(x,dy) \tag{6}$$

is a regular conditional distribution of $A*B$.

3 Maximal d_∞-asymmetric copulas and their extent of dependence with respect to ζ_1 and ξ

Since ordinal sums will play an important role in what follows, we briefly recall their definition. We follow [6] and let $I \subseteq \mathbb{N}$ be some finite index set, $((a_i,b_i))_{i\in I}$ be a family of non-overlapping intervals with $0 \le a_i < b_i \le 1$ for each $i \in I$ such that $\bigcup_{i\in I}[a_i,b_i] = [0,1]$ holds. Furthermore, $(C_i)_{i\in I}$ denotes a family of bivariate copulas. Then the copula C defined by

$$C(x,y) = \begin{cases} a_i + (b_i - a_i)C_i\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right), & (x,y) \in (a_i,b_i)^2 \\ \min\{x,y\} & \text{elsewhere} \end{cases}$$

is an ordinal sum, and we write $C = (\langle a_i, b_i, C_i\rangle)_{i\in I}$. The following lemma gathers some useful formulas for D_1 and D_2^2 which will be used in the sequel.

Lemma 3.1. *Let $C = (\langle a_i, b_i, C_i\rangle)_{i\in I}$ be an ordinal sum with $I := \{1,\dots,n\}$ for some $n \in \mathbb{N}$. Then*

$$D_2^2(C,\Pi) = \sum_{i=1}^n \left((b_i-a_i)^2 D_2^2(C_i,\Pi)\right) + f(a_1,\dots,a_n,b_1,\dots,b_n),$$

$$D_1(C,\Pi) = \sum_{i=1}^n \left((b_i-a_i)^2 \int_{[0,1]}\int_{[0,1]} \left|K_{C_i}(x,[0,y]) - (a_i + (b_i-a_i)y)\right| d\lambda(x)d\lambda(y)\right) + g(a_1,\dots,a_n,b_1,\dots,b_n),$$

whereby f and g are given by $f(a_1,\dots,a_n,b_1,\dots,b_n) := \sum_{i=1}^n \left(\frac{(b_i-a_i)^2}{3} + (b_i-a_i)(1-b_i)\right) - \frac{1}{3}$ and $g(a_1,\dots,a_n,b_1,\dots,b_n) := \sum_{i=1}^n (b_i-a_i)\left(\frac{1}{2} - b_i + \frac{b_i^2}{2} + \frac{a_i^2}{2}\right)$, respectively.

Proof. The definition of D_2^2 yields

$$\begin{aligned} D_2^2(C,\Pi) &= \int_{[0,1]}\int_{[0,1]} \left(K_C(x,[0,y]) - K_\Pi(x,[0,y])\right)^2 d\lambda(x)d\lambda(y) \\ &= \int_{[0,1]}\int_{[0,1]} K_C(x,[0,y])^2 d\lambda(x)d\lambda(y) \\ &\quad - 2\int_{[0,1]} y \int_{[0,1]} K_C(x,[0,y]) d\lambda(x)d\lambda(y) + \int_{[0,1]}\int_{[0,1]} y^2 d\lambda(x)d\lambda(y) \\ &= \int_{[0,1]}\int_{[0,1]} K_C(x,[0,y])^2 d\lambda(x)d\lambda(y) - \frac{1}{3} \end{aligned}$$

for every $C \in \mathcal{C}$. Using the fact that (without loss of generality) the Markov kernel $K_C(x,[0,y])$ of C is 0 below the squares $(a_i,b_i)^2$ and 1 above $(a_i,b_i)^2$, and applying change of coordinates yields

$$\begin{aligned} D_2^2(C,\Pi) + \frac{1}{3} &= \sum_{i=1}^n \left(\int_{(a_i,b_i)}\int_{(a_i,b_i)} K_C(x,[0,y])^2 d\lambda(x)d\lambda(y) + \int_{(b_i,1)}\int_{(a_i,b_i)} 1 d\lambda(x)d\lambda(y) \right) \\ &= \sum_{i=1}^n \left((b_i - a_i)^2 \int_{[0,1]}\int_{[0,1]} K_{C_i}(x,[0,y])^2 d\lambda(x)d\lambda(y) + (b_i - a_i)(1 - b_i) \right) \\ &= \sum_{i=1}^n \left((b_i - a_i)^2 \left(D_2^2(C_i,\Pi) + \frac{1}{3} \right) + (b_i - a_i)(1 - b_i) \right). \end{aligned}$$

Analogously we get

$$\begin{aligned} D_1(C,\Pi) &= \sum_{i=1}^n \int_{[0,1]}\int_{(a_i,b_i)} |K_C(x,[0,y] - y| d\lambda(x)d\lambda(y) \\ &= \sum_{i=1}^n \int_{(a_i,b_i)}\int_{(a_i,b_i)} \left| K_{C_i}\left(\frac{x - a_i}{b_i - a_i}, \left[0, \frac{y - a_i}{b_i - a_i}\right] \right) - y \right| d\lambda(x)d\lambda(y) \\ &\quad + \sum_{i=1}^n \int_{(b_i,1)}\int_{(a_i,b_i)} (1 - y)\, d\lambda(x)d\lambda(y) + \sum_{i=1}^n \int_{(0,a_i)}\int_{(a_i,b_i)} y\, d\lambda(x)d\lambda(y) \\ &= \sum_{i=1}^n \left((b_i - a_i)^2 \int_{[0,1]}\int_{[0,1]} \big| K_{C_i}(x,[0,y]) - (a_i + (b_i - a_i)y) \big| d\lambda(x)d\lambda(y) \right) \\ &\quad + g(a_1,\dots,a_n,b_1,\dots,b_n), \end{aligned}$$

with $g(a_1,\dots,a_n,b_1,\dots,b_n)$ as in the theorem. □

As a direct consequence the dependence measure ξ of ordinal sums can easily be expressed in terms of $\xi(C_i)$:

Corollary 3.2. *Let $C = (\langle a_i, b_i, C_i \rangle)_{i \in I}$ be an ordinal sum with $I := \{1,\dots,n\}$ for some $n \in \mathbb{N}$. Then*

$$\xi(C) = \sum_{i=1}^n (b_i - a_i)^2 \xi(C_i) + 6 f(a_1,\dots,a_n,b_1,\dots,b_n)$$

holds, where f is defined according to Lemma 3.1 and only depends on the partition.

The following example shows that ordinal sums can be used to construct copulas attaining every possible dependence value w.r.t. to ξ and ζ_1.

Example 3.3. Consider $C_s = (\langle a_i, b_i, C_i \rangle)_{i\in\{1,2\}}$, whereby $a_1 := 0, a_2 := s$, $b_1 := s$ and $b_2 := 1$ for $s \in [0,1]$ and set $C_1 = \Pi$ as well as $C_2 = M$. Figure 1 depicts the support of μ_{C_s} for different choices of $s \in [0,1]$. Using Corollary 3.2 we have $\xi(C_s) = (1-s)^2 + 6\left(\frac{s^2}{3} + s(1-s) + \frac{(1-s)^2}{3} - \frac{1}{3}\right) = 1 - s^2$. Therefore, the map $\varphi : [0,1] \to [0,1]$ defined by $s \mapsto \xi(C_s)$ is continuous and onto. The same holds for $\zeta_1(C_s) = 1 - s^3$.

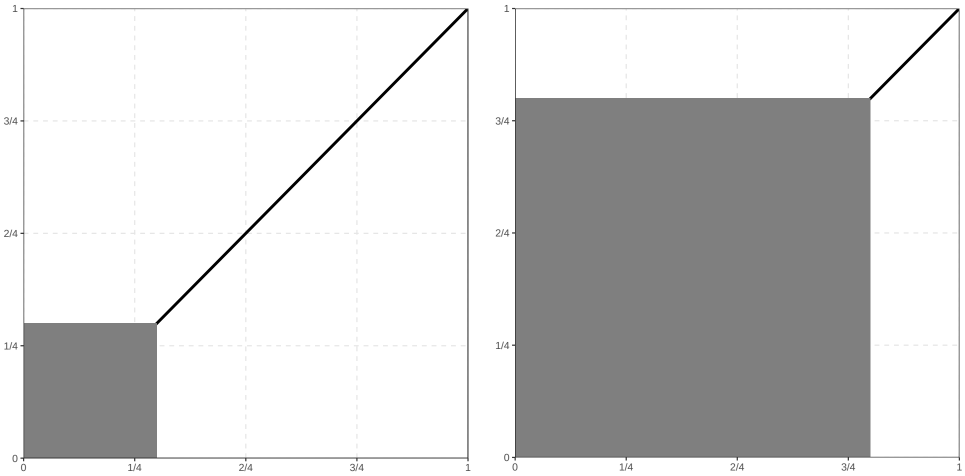

Figure 1: Mass distribution of the doubly stochastic measure μ_{C_s} for $s = 0.3$ (left panel) and $s = 0.8$ (right panel) considered in Example 3.3. For the dependence measures ξ and ζ_1 we get $\xi(C_{0.3}) = 0.91$ and $\xi(C_{0.8}) = 0.36$ as well as $\zeta_1(C_{0.3}) = 0.973$ and $\zeta_1(C_{0.8}) = 0.488$.

Before deriving some first results concerning the range of the dependence measures $\xi(A)$ and $\zeta_1(A)$ for maximal d_∞-asymmetric copulas A, we recall the characterizations of maximal d_∞-asymmetry derived in [21, 15]: $d_\infty(A, A^t)$ is maximal if and only if $A\left(\frac{2}{3}, \frac{1}{3}\right) = 0$ and $A\left(\frac{1}{3}, \frac{2}{3}\right) = \frac{1}{3}$ or $A^t\left(\frac{2}{3}, \frac{1}{3}\right) = 0$ and $A^t\left(\frac{1}{3}, \frac{2}{3}\right) = \frac{1}{3}$. Without loss of generality we may focus on the case $A\left(\frac{1}{3}, \frac{2}{3}\right) = \frac{1}{3}$ and $A\left(\frac{2}{3}, \frac{1}{3}\right) = 0$. Since A is doubly stochastic in this case we obviously have $\mu_A([0, \frac{1}{3}] \times [\frac{1}{3}, \frac{2}{3}]) = \mu_A([\frac{1}{3}, \frac{2}{3}] \times [\frac{2}{3}, 1]) = \mu_A([\frac{2}{3}, 1] \times [0, \frac{1}{3}]) = \frac{1}{3}$. As a direct consequence we can find copulas $A_1, A_2, A_3 \in \mathcal{C}$ fulfilling

$$\mu_A = \frac{1}{3}\mu_{A_1}^{f_{12}} + \frac{1}{3}\mu_{A_2}^{f_{23}} + \frac{1}{3}\mu_{A_3}^{f_{31}}, \tag{7}$$

whereby the functions $f_{ij} : [0,1]^2 \to \left[\frac{i-1}{3}, \frac{i}{3}\right] \times \left[\frac{j-1}{3}, \frac{j}{3}\right]$ are given by $f_{ij}(x,y) = \left(\frac{x+i-1}{3}, \frac{y+j-1}{3}\right)$ for each $(i,j) \in \{1,2,3\}^2$ (and $\mu_A^{f_{ij}}$ denotes the push-forward of μ_A via f_{ij}).

Theorem 3.4. *If $A \in \mathcal{C}$ has maximal d_∞-asymmetry, i.e., if $\delta(A) = 3d_\infty(A, A^t) = 1$ holds, then ξ satisfies $\xi(A) \in \left[\frac{2}{3}, 1\right]$. Moreover, for every $s \in \left[\frac{2}{3}, 1\right]$ there exists a copula A with $\delta(A) = 1$ fulfilling $\xi(A) = s$.*

Proof. We may assume that $A\left(\frac{1}{3}, \frac{2}{3}\right) = \frac{1}{3}$ and $A\left(\frac{2}{3}, \frac{1}{3}\right) = 0$. Then there exist copulas $A_1, A_2, A_3 \in$

$\mathcal{C}$ such that $\mu_A = \frac{1}{3}\mu_{A_2}^{f_{12}} + \frac{1}{3}\mu_{A_3}^{f_{23}} + \frac{1}{3}\mu_{A_1}^{f_{31}}$ holds. Defining $h : [0,1] \to [0,1]$ by

$$h(x) = \begin{cases} \frac{1}{3} + x & \text{if } x \in \left[0, \frac{2}{3}\right] \\ x - \frac{2}{3} & \text{if } x \in \left(\frac{2}{3}, 1\right], \end{cases}$$

we have $h \in \mathcal{T}_b$ and $\mathcal{S}_h(A) = \left(\langle \frac{i-1}{3}, \frac{i}{3}, A_i \rangle\right)_{i \in \{1,2,3\}}$. Applying Lemma 2.1 and 3.2 we therefore obtain

$$\xi(A) = \xi(\mathcal{S}_h(A)) = 6f(a_1, \ldots, a_n, b_1, \ldots, b_n) + \sum_{i=1}^{3} \frac{1}{9}\xi(A_i) = \frac{2}{3} + \sum_{i=1}^{3} \frac{1}{9}\xi(A_i) \geq \frac{2}{3},$$

with equality if and only if $A_i = \Pi$ for every $i = 1, 2, 3$.
Defining A_s by $\mu_{A_s} := \frac{1}{3}\mu_{C_s}^{f_{12}} + \frac{1}{3}\mu_{C_s}^{f_{23}} + \frac{1}{3}\mu_{C_s}^{f_{31}}$ with C_s as in Example 3.3 yields

$$\xi(A_s) = \xi(\mathcal{S}_h(A_s)) = \frac{1}{3}\xi(C_s) + \frac{2}{3}.$$

Considering $\xi(C_s) = 1 - s^2 \in [0,1]$ for $s \in [0,1]$ and using the same arguments as in Example 3.3 it follows that for every $s_0 \in \left[\frac{2}{3}, 1\right]$ we find a copula $A \in \mathcal{C}$ with $\xi(A) = s_0$ and $3d_\infty(A, A^t) = \delta(A) = 1$. □

Since ζ_1 and ξ are similar by construction, one might expect the analogous statements for ζ_1. Notice, however, that a different proof is needed since according to Lemma 3.1 the formulas for D_1 are more involved.

Theorem 3.5. *If $A \in \mathcal{C}$ has maximal d_∞-asymmetry, then ζ_1 satisfies $\zeta_1(A) \in [\frac{5}{6}, 1]$. Furthermore, for every $s \in [\frac{5}{6}, 1]$ there exists a copula A with $\delta(A) = 1$ fulfilling $\zeta_1(A) = s$.*

Proof. Proceeding as in the proof of Theorem 3.4 we obtain $\mathcal{S}_h(A) = \left(\langle \frac{i-1}{3}, \frac{i}{3}, A_i \rangle\right)_{i \in \{1,2,3\}}$. Considering the (SI)-rearrangement $\mathcal{S}_h(A)^\uparrow$ of $\mathcal{S}_h(A)$ it is clear that $\mathcal{S}_h(A)^\uparrow$ is an ordinal sum again and can be expressed as $\mathcal{S}_h(A)^\uparrow = \left(\left\langle \frac{i-1}{3}, \frac{i}{3}, A_i^\uparrow \right\rangle\right)_{i \in \{1,2,3\}}$. Since every $A_i^\uparrow$ is stochastically increasing (SI) and hence fulfills $A_i^\uparrow(x,y) \geq \Pi^\uparrow(x,y) = \Pi(x,y)$ for every $(x,y) \in [0,1]^2$ and every $i \in \{1,2,3\}$ (see, e.g., [22][Section 5.2]), we obtain that

$$\mathcal{S}_h(A)^\uparrow \geq C_\Pi := \left(\left\langle \frac{i-1}{3}, \frac{i}{3}, \Pi \right\rangle\right)_{i \in \{1,2,3\}}$$

holds pointwise. Due to the fact that ζ_1 is monotone w.r.t. the pointwise order on $\mathcal{C}^\uparrow$ and ζ_1 is invariant w.r.t. to (SI)-rearrangements (see [26]), we get

$$\zeta_1(A) = \zeta_1(\mathcal{S}_h(A)) = \zeta_1(\mathcal{S}_h(A)^\uparrow) \geq \zeta_1(C_\Pi) = \frac{5}{6},$$

where the last equality follows from Lemma 3.1 (the detailed calculations are deferred to the Appendix 6). To show the second assertion we can proceed analogously to the proof of Theorem 3.4 and use shrunk copies of the copula C_s defined in Example 3.3 (see Appendix 6). □

While the minimum value of ξ for a copula $A \in \mathcal{C}$ with maximal d_∞-asymmetry is attained if and only if $A_i = \Pi$ for every $i = 1, 2, 3$ in Eq. (7), ζ_1 exhibits a different behaviour as demonstrated in the following example:

Example 3.6. Let $A_1 \in \mathcal{C}^{\uparrow}$ be defined by

$$A_1(x,y) = xy + \frac{1}{2}x(1-x)y(1-y).$$

Then a version of the corresponding Markov kernel of A_1 is given by $K_{A_1}(x,[0,y]) = y + \frac{1}{2}(2x-1)y(y-1)$. Furthermore, we set $A_3 = A_1$ and $A_2 = \Pi$ and let A denote the ordinal sum given by $A := \left(\langle \frac{i-1}{3}, \frac{i}{3}, A_i\rangle\right)_{i\in\{1,2,3\}}$ and C_Π be the ordinal sum given by $C_\Pi := \left(\langle \frac{i-1}{3}, \frac{i}{3}, \Pi\rangle\right)_{i\in\{1,2,3\}}$ (see Figure 2). By construction we have $A \neq C_\Pi$, however, considering

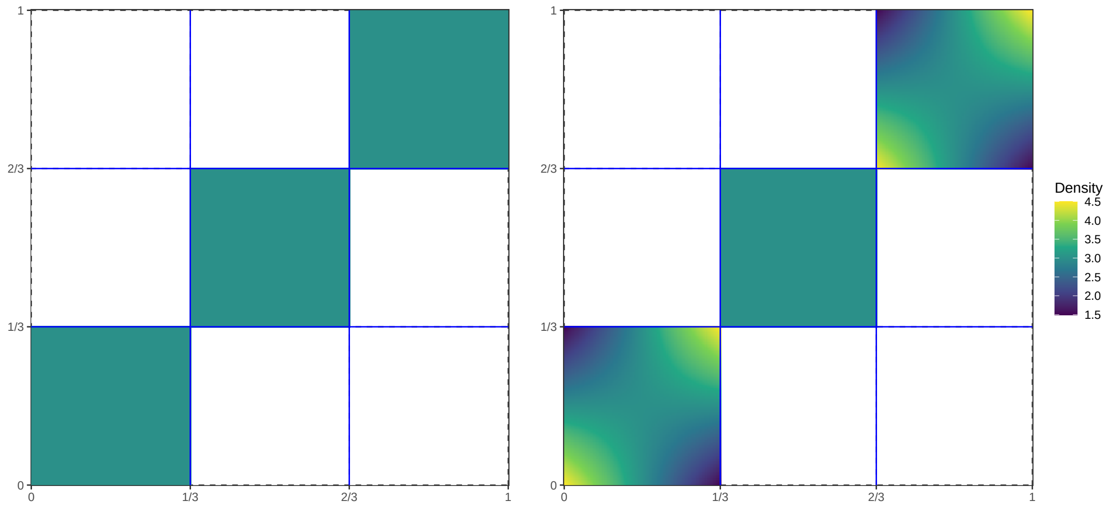

Figure 2: Density of the stochastically increasing copulas C_Π (left panel) and A (right panel) considered in Example 3.6. Although $A(x,y) \geq C_\Pi(x,y)$ holds for every $(x,y) \in [0,1]^2$ and there exists some (x,y) with $A(x,y) > C_\Pi(x,y)$ we have $\zeta_1(A) = \zeta_1(C_\Pi)$. On the contrary, ξ fulfills $\xi(A) > \xi(C_\Pi)$.

$$\int_{[0,1]}\int_{[0,1]} \frac{1}{2}(2x-1)y(y-1)d\lambda(x)d\lambda(y) = 0,$$

yields

$$\begin{aligned}
\int_{[0,1]}\int_{[0,1]} \left|K_\Pi(x,[0,y]) - \frac{y}{3}\right| d\lambda(x)d\lambda(y) &= \int_{[0,1]}\int_{[0,1]} \frac{2y}{3} d\lambda(x)d\lambda(y) \\
&= \int_{[0,1]}\int_{[0,1]} \left(\frac{2y}{3} + \frac{1}{2}(2x-1)y(y-1)\right) d\lambda(x)d\lambda(y) \\
&= \int_{[0,1]}\int_{[0,1]} \left|\frac{2y}{3} + \frac{1}{2}(2x-1)y(y-1)\right| d\lambda(x)d\lambda(y) \\
&= \int_{[0,1]}\int_{[0,1]} \left|K_{A_1}(x,[0,y]) - \frac{y}{3}\right| d\lambda(x)d\lambda(y)
\end{aligned}$$

and analogously we get

$$\int_{[0,1]}\int_{[0,1]} \left|K_\Pi(x,[0,y]) - \left(\frac{2}{3}+\frac{y}{3}\right)\right| d\lambda(x)d\lambda(y) = \int_{[0,1]}\int_{[0,1]} \left|K_{A_1}(x,[0,y]) - \left(\frac{2}{3}+\frac{y}{3}\right)\right| d\lambda(x)d\lambda(y).$$

Applying Lemma 3.1 we obtain $\zeta_1(A) = \zeta_1(C_\Pi) = \frac{5}{6}$.

Remark 3.7. Considering the monotonicity of ζ_1 with respect to the pointwise order on $\mathcal{C}^\uparrow$ as proved in [26], Example 3.6 shows that there exist copulas $A, B \in \mathcal{C}^\uparrow$ with $A \leq B$ pointwise and $A(x,y) < B(x,y)$ for some $(x,y) \in [0,1]^2$ fulfilling $\zeta_1(A) = \zeta_1(B)$.

4 Maximal D_1-asymmetry of copulas revisited

In this section we complement characterizations of copulas with maximal D_1-asymmetry going back to [13] and derive some topological properties of subclasses. To be consistent with the notation in [13], the family of copulas with maximal D_1-asymmetry is denoted by

$$\mathcal{C}^{\kappa=1} := \left\{A \in \mathcal{C} : \kappa(A) := 2D_1(A, A^t) = 1\right\} \subseteq \mathcal{C},$$

the subclass of mutually completely dependent copulas is denoted by $\mathcal{C}^{\kappa=1}_{mcd}$. We start with the family of mutually completely dependent copulas and show closedness w.r.t. the metric D_∂.

Proposition 4.1. *The set $\mathcal{C}^{\kappa=1}_{mcd}$ is closed in $(\mathcal{C}, D_\partial)$.*

Proof. Let $(A_{h_n})_{n\in\mathbb{N}}$ be a sequence of mutually completely dependent copulas with D_∂-limit A. Since according to [27] the family of completely dependent copulas is closed w.r.t. D_1 we obtain $A \in \mathcal{C}_{cd}$ and $A^t \in \mathcal{C}_{cd}$. Using [27, Lemma 10] there exist λ-preserving transformations $g, g' \in \mathcal{T}$ such that a version of the Markov kernel $K_A(\cdot,\cdot)$ and $K_{A^t}(\cdot,\cdot)$ is given by $K_A(x,E) = \mathbb{1}_E(g(x))$ and $K_{A^t}(x,E) = \mathbb{1}_E(g'(x))$, respectively. Furthermore, since a copula A is completely dependent if and only if it is left-invertible w.r.t. the $*$-product (see [27]) we have $M = A^t * A$. Applying Lemma 2.2 therefore yields that $g \circ g'(x) = id(x)$ for λ-a.e. $x \in [0,1]$. Using the fact that g is surjective λ-almost everywhere, there exists a λ-preserving and bijective transformation $h \in \mathcal{T}_b$ such that $h = g$ holds λ-a.e., implying $A = A_h \in \mathcal{C}_{mcd}$. It remains to show that $D_1(A_h, A_h^t) = \frac{1}{2}$, which can be done as follows. Using [13][Theorem 3.5] and the triangle inequality we get

$$\frac{1}{2} = \int_{[0,1]} \left|h_n - h_n^{-1}\right| d\lambda(x) \leq \int_{[0,1]} |h_n - h|\, d\lambda(x) + \int_{[0,1]} \left|h - h^{-1}\right| d\lambda(x) + \int_{[0,1]} \left|h^{-1} - h_n^{-1}\right| d\lambda(x)$$

for every $n \in \mathbb{N}$. Applying [27][Proposition 15 (ii)] yields

$$D_1(A_h, A_h^t) = \int_{[0,1]} \left|h(x) - h^{-1}(x)\right| d\lambda(x) \geq \frac{1}{2} - \left[D_1(A_{h_n}, A_h) + D_1(A_{h_n}^t, A_h^t)\right] = \frac{1}{2} - D_\partial(A_{h_n}, A_h).$$

Together with the fact that the maximal distance can not exceed $\frac{1}{2}$ it follows that $D_1(A_h, A_h^t) = \frac{1}{2}$, which completes the proof. □

The following example shows that the set $\mathcal{C}^{\kappa=1}_{mcd}$ is not closed w.r.t. the metric D_1.

Example 4.2. Let $A_{h_n} \in \mathcal{C}_{mcd}$ be the mutually completely dependent copula induced by the bijective measure-preserving transformation $h_n \colon [0,1] \to [0,1]$, given by

$$h_n(x) = \begin{cases} x + \frac{j-1}{n} & \text{if } x \in \left[\frac{j-1}{n}, \frac{j}{n}\right) \text{ and } j \in \left\{1, \dots, \frac{n}{2}\right\} \\ x - 1 + \frac{j}{n} & \text{if } x \in \left[\frac{j-1}{n}, \frac{j}{n}\right) \text{ and } j \in \left\{\frac{n}{2}+1, \dots, n\right\} \\ 1 & \text{if } x = 1 \end{cases}$$

for all $n \in 2\mathbb{N}$ and let $A_h \in \mathcal{C}_{cd}$ be the completely dependent copula induced by the λ-preserving transformation $h : [0,1] \to [0,1]$ given by $h(x) := 2x(mod 1)$ (see Figure 1 in [9]). Setting $C_n := \big(\langle \frac{i-1}{4}, \frac{i}{4}, A_{h_{2n}} \rangle\big)_{i\in\{1,2,3,4\}}$ and $C := \big(\langle \frac{i-1}{4}, \frac{i}{4}, A_h \rangle\big)_{i\in\{1,2,3,4\}}$ we have $C_n \in \mathcal{C}_{mcd}$ and $C \in \mathcal{C}_{cd}$ and according to [9, Example 3.3] it is straightforward to verify that $\lim_{n\to\infty} D_1(C_n, C) = 0$. As next step we reorder the shrunk copulas to obtain maximal D_1-asymmetry. Let f denote the λ-preserving interval exchange transformation $f : [0,1] \to [0,1]$ defined by $f(x) := (x - \frac{1}{4})\mathbb{1}_{\left(\frac{1}{4},1\right]}(x) + (x + \frac{3}{4})\mathbb{1}_{\left[0,\frac{1}{4}\right]}(x)$ and, furthermore, let $\mathcal{S}_f(C_n) \in \mathcal{C}_{mcd}$ and $\mathcal{S}_f(C) \in \mathcal{C}_{cd}$ denote the respective shuffles (see Figure 3). Due to the fact that the metric D_1 is shuffle-invariant w.r.t. bijective transformations (using the same arguments as in the proof of Lemma 2.1) yields

$$\lim_{n\to\infty} D_1(\mathcal{S}_f(C_n), \mathcal{S}_f(C)) = \lim_{n\to\infty} D_1(C_n, C) = 0.$$

Setting $U := \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$ and considering property (3) of Theorem 4.1 in [13] (see also Theorem 4.4 *(iii)* in the sequel) we directly obtain that $\mathcal{S}_f(C_n)$ and $\mathcal{S}_f(C)$ are maximal asymmetric w.r.t. D_1, which shows that $\mathcal{C}_{mcd}^{\kappa=1}$ is not closed w.r.t. the metric D_1.

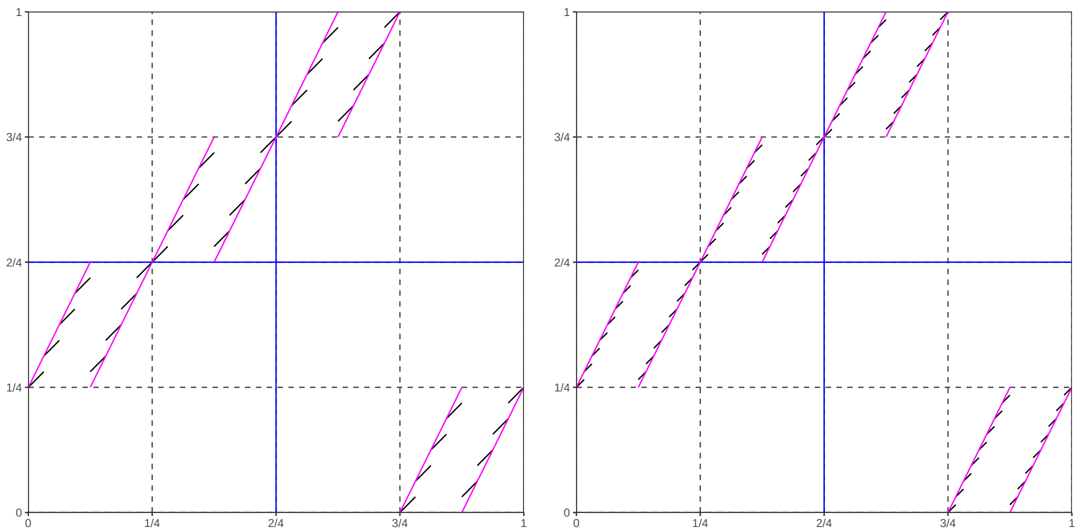

Figure 3: The support of the copulas $\mu_{\mathcal{S}_f(C_n)}$ (black) for $n = 4$ (left panel) and $n = 8$ (right panel) as well as the copula $\mu_{\mathcal{S}_f(C)}$ (magenta) as considered in Example 4.2.

Leaving the subclass of mutually completely dependent copulas we will now derive novel and handy characterizations of copulas with maximal D_1-asymmetry and then show some topological properties. The following lemma, showing that the $*$-product can not increase the D_p-distance, will be useful in the sequel. The result has already been stated for D_1 in a slightly different context in [28].

Lemma 4.3. *For every $A, B, C \in \mathcal{C}$ the following inequality holds for every $p \in [1, \infty)$:*

$$D_p^p(A * B, A * C) \leq D_p^p(B, C).$$

Proof. Applying Lemma 2.2, Jensen's inequality, disintegration and using the fact that μ_A is doubly stochastic we obtain

$$
\begin{aligned}
D_p^p(A*B, A*C) &= \int_{[0,1]}\int_{[0,1]}\left|\int_{[0,1]} K_B(t,[0,y])K_A(x,dt) - \int_{[0,1]} K_C(t,[0,y])K_A(x,dt)\right|^p d\lambda(x)d\lambda(y) \\
&\le \int_{[0,1]}\int_{[0,1]}\int_{[0,1]} |K_B(t,[0,y]) - K_C(t,[0,y])|^p K_A(x,dt)d\lambda(x)d\lambda(y) \\
&= \int_{[0,1]}\int_{[0,1]^2} |K_B(t,[0,y]) - K_C(t,[0,y])|^p d\mu_A(x,t)d\lambda(y) \\
&= \int_{[0,1]}\int_{[0,1]} |K_B(t,[0,y]) - K_C(t,[0,y])|^p d\lambda(t)d\lambda(y) = D_p^p(B,C),
\end{aligned}
$$

which completes the proof. □

The next theorem gathers several equivalent characterizations of copulas having maximal D_1-asymmetry (see [13]), the novel ones established here are *(v)* and *(vi)*.

Theorem 4.4. *For every $A \in \mathcal{C}$ the following statements are equivalent:*

(i) $\kappa(A) = 1$,

(ii) $\Phi_{A,A^t}(\frac{1}{2}) = 1$ *(or equivalently, A has maximal D_∞-asymmetry),*

(iii) *there exists a Borel set $U \in \mathcal{B}([0,1])$ with the following properties:*

$$
\lambda(U \cap [0,\tfrac{1}{2}]) = \lambda(U \cap [\tfrac{1}{2},1]) = \tfrac{1}{4},\ \ \mu_A(U \times [0,\tfrac{1}{2}]) = \tfrac{1}{2},\ \ \mu_A([0,\tfrac{1}{2}] \times U) = 0,
$$

(iv) *there exist sets $U_1, U_2 \in \mathcal{B}([0,1])$ with $U_1 \subseteq [0,\frac{1}{2}]$, $U_2 \subseteq (\frac{1}{2},1]$, $\lambda(U_1) = \lambda(U_2) = \frac{1}{4}$ and $V_1 := [0,\frac{1}{2}] \setminus U_1$ and $V_2 := (\frac{1}{2},1] \setminus U_2$, and copulas $C_1, C_2, C_3, C_4 \in \mathcal{C}$ such that the following identity*

$$
A(x,y) = \frac{1}{4}\left[C_1(F_1(x),G_1(y)) + C_2(G_1(x),G_2(y)) + C_3(F_2(x),F_1(y)) + C_4(G_2(x),F_2(y))\right]
$$

holds, whereby $F_i(x) := 4\lambda(U_i \cap [0,x])$ and $G_i(x) := 4\lambda(V_i \cap [0,x])$ for $i = 1,2$.

(v) $(A*A)\left(\frac{1}{2},\frac{1}{2}\right) = 0$,

(vi) $D_1(A*A, A*A^t) = \frac{1}{2}$.

Proof. The equivalences of *(i)*, *(ii)*, *(iii)* and *(iv)* have already been proved in [13]. Note that the equivalence in property *(ii)* directly follows from the facts that Φ_{A,A^t} is Lipschitz continuous with Lipschitz constant 2 and the function $\Phi_{A,A^t} : [0,1] \to [0,1]$ fulfills $\Phi_{A,A^t}(y) \le \min\{2y, 2(1-y)\}$ for every $y \in [0,1]$ (see Lemma 5 in [27]). To show that *(i)* and *(v)* are equivalent we may proceed as follows: Suppose that $\kappa(A) = 1$. Then according to property *(iii)* there exists a Borel set $U \in \mathcal{B}([0,1])$ with $\lambda(U) = \frac{1}{2}$ such that $K_A(x,[0,\frac{1}{2}]) = 1$ for every $x \in U$. Applying Eq. (2) and disintegration yields another Borel set $V \subseteq U^c$ with $\lambda(V) = \frac{1}{2}$ and $K_A(x,[0,\frac{1}{2}]) = 0$ for every $x \in V$. Setting $\tilde{V} := U^c \setminus V$, then obviously $\lambda(\tilde{V}) = 0$ holds, and applying Lemma 2.2 we obtain

$$
\begin{aligned}
\mu_{A*A}([0,\tfrac{1}{2}] \times [0,\tfrac{1}{2}]) &= \int_{[0,\frac{1}{2}]}\int_{[0,1]} K_A(s,[0,\tfrac{1}{2}])K_A(x,ds)d\lambda(x) \\
&= \int_{[0,\frac{1}{2}]}\left(\int_U 1\,K_A(x,ds) + \int_V 0\,K_A(x,ds) + \int_{\tilde{V}} K_A(s,[0,\tfrac{1}{2}])K_A(x,ds)\right)d\lambda(x)
\end{aligned}
$$

$$\leq \int_{[0,\frac{1}{2}]} K_A(x,U)d\lambda(x) + \int_{[0,\frac{1}{2}]} K_A(x,\tilde{V})d\lambda(x)$$
$$\leq \mu_A\left([0,\tfrac{1}{2}]\times U\right) + \mu_A\left([0,1]\times \tilde{V}\right) = 0 + \lambda(\tilde{V}) = 0.$$

Suppose now that $(A*A)(\frac{1}{2},\frac{1}{2}) = 0$ holds. Then according to Eq. (5) we have

$$\int_{[0,1]} K_{A^t}(x,[0,\tfrac{1}{2}])K_A(x,[0,\tfrac{1}{2}])d\lambda(x) = 0,$$

so there exists a set $\Lambda \in \mathcal{B}([0,1])$ with $\lambda(\Lambda) = 1$ such that $K_{A^t}(x,[0,\frac{1}{2}])K_A(x,[0,\frac{1}{2}]) = 0$ holds for all $x \in \Lambda$. Considering $\min\{a,b\} = \frac{1}{2}(a+b-|a-b|)$ therefore yields

$$\begin{aligned}
\Phi_{A,A^t}(\tfrac{1}{2}) &= \int_{[0,1]} |K_A(x,[0,\tfrac{1}{2}]) - K_{A^t}(x,[0,\tfrac{1}{2}])|d\lambda(x) \\
&= \int_{[0,1]} K_A(x,[0,\tfrac{1}{2}])d\lambda(x) + \int_{[0,1]} K_{A^t}(x,[0,\tfrac{1}{2}])d\lambda(x) \\
&\quad - 2\int_{[0,1]} \min\{K_A(x,[0,\tfrac{1}{2}]), K_{A^t}(x,[0,\tfrac{1}{2}])\}d\lambda(x) \\
&= \tfrac{1}{2} + \tfrac{1}{2} - 2\int_{\Lambda} \min\{K_A(x,[0,\tfrac{1}{2}]), K_{A^t}(x,[0,\tfrac{1}{2}])\}d\lambda(x) = \frac{1}{2} + \frac{1}{2} - 2\cdot 0 = 1.
\end{aligned}$$

To show the equivalence of *(i)* and *(vi)* first assume that $D_1(A*A, A*A^t) = \frac{1}{2}$. Then applying Lemma 4.3 directly yields $D_1(A,A^t) \geq \frac{1}{2}$, hence $\kappa(A) = 1$. On the other hand, if $\kappa(A) = 1$ holds we may proceed as follows: There exists a Borel set $U \in \mathcal{B}([0,1])$ with $\lambda(U) = \frac{1}{2}$ and $K_A(x,[0,\frac{1}{2}]) = 1$ as well as $K_{A^t}(x,[0,\frac{1}{2}]) = 0$ for every $x \in U$. Using disintegration and Eq. (2) there exists a Borel set $V \subseteq U^c$ with $\lambda(V) = \frac{1}{2}$ and $K_A(x,[0,\frac{1}{2}]) = 0$ and $K_{A^t}(x,[0,\frac{1}{2}]) = 1$ for every $x \in V$. As before, set $\tilde{V} := U^c \setminus V$. Applying Lemma 2.2 yields

$$\begin{aligned}
A*A^t\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \mu_{A*A^t}\left([0,\tfrac{1}{2}]\times[0,\tfrac{1}{2}]\right) &= \int_{[0,\frac{1}{2}]}\int_{[0,1]} K_{A^t}\left(s,[0,\tfrac{1}{2}]\right)K_A(x,ds)d\lambda(x) \\
&\leq \int_{[0,\frac{1}{2}]}\int_{U^c} 1\, K_A(x,ds)d\lambda(x) = \int_{[0,\frac{1}{2}]} K_A(x,U^c)d\lambda(x) \\
&= \mu_A\left([0,\tfrac{1}{2}]\times U^c\right) = \frac{1}{2} - \mu_A\left([0,\tfrac{1}{2}]\times U\right) = \frac{1}{2}
\end{aligned}$$

as well as

$$\begin{aligned}
\mu_{A*A^t}\left([0,\tfrac{1}{2}]\times[0,\tfrac{1}{2}]\right) &= \int_{[0,\frac{1}{2}]}\left(\int_U 0\, K_A(x,ds) + \int_V 1\, K_A(x,ds) + \int_{\tilde{V}} K_{A^t}\left(s,[0,\tfrac{1}{2}]\right)K_A(x,ds)\right)d\lambda(x) \\
&\geq \int_{[0,\frac{1}{2}]} K_A(x,V)d\lambda(x) = \mu_A\left([0,\tfrac{1}{2}]\times V\right) = \tfrac{1}{2} - \mu_A\left([0,\tfrac{1}{2}]\times V^c\right) \\
&= \tfrac{1}{2} - \left(\mu_A\left([0,\tfrac{1}{2}]\times U\right) + \mu_A\left([0,\tfrac{1}{2}]\times\tilde{V}\right)\right) \geq \frac{1}{2}.
\end{aligned}$$

Together with property *(v)* there exist Borel sets $\Delta_1 \subseteq [0,\frac{1}{2}]$ and $\Delta_2 \subseteq (\frac{1}{2},1]$ with $\lambda(\Delta_1) = \lambda(\Delta_2) = \frac{1}{2}$ such that

$$K_{A*A}\left(x_1,[0,\tfrac{1}{2}]\right) = 0,\ K_{A*A}\left(x_2,[0,\tfrac{1}{2}]\right) = 1 \text{ and } K_{A*A^t}\left(x_1,[0,\tfrac{1}{2}]\right) = 1,\ K_{A*A^t}\left(x_2,[0,\tfrac{1}{2}]\right) = 0$$

for every $x_1 \in \Delta_1$ and $x_2 \in \Delta_2$, which gives

$$\Phi_{A*A,A*A^t}\left(\tfrac{1}{2}\right) = \int_{[0,1]} \left|K_{A*A}\left(x,\left[0,\tfrac{1}{2}\right]\right) - K_{A*A^t}\left(x,\left[0,\tfrac{1}{2}\right]\right)\right| d\lambda(x) = \lambda(\Delta_1) + \lambda(\Delta_2) = 1.$$

Since $y \mapsto \Phi_{A,B}(y)$ is Lipschitz continuous with Lipschitz constant 2 (see [27][Lemma 5]), the property that $D_1(A * A, A * A^t) = \frac{1}{2}$ follows immediately and the proof is complete. □

Remark 4.5. For mutually completely dependent copulas $A_h \in \mathcal{C}_{mcd}$, property *(vi)* of Theorem 4.4 simplifies to

$$\kappa(A_h) = 1 \text{ if and only if } D_1(A_h * A_h, M) = D_1(A_{h^2}, M) = \|h^2 - id\|_1 = \tfrac{1}{2},$$

where the second equality of the right hand side directly follows from [27][Proposition 15 (ii)]. Furthermore, considering property *(v)* of Theorem 4.4 one might conjecture that $(A*A^t)(\frac{1}{2},\frac{1}{2}) = \frac{1}{2}$ also implies $\kappa(A) = 1$. For the symmetric copula M, however, it is clear that $M\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1}{2}$ as well as $M * M^t = M * M = M$ holds.

Not surprisingly, the following result holds.

Proposition 4.6. *The set $\mathcal{C}^{\kappa=1}$ is closed in $(\mathcal{C}, D_\partial)$.*

Proof. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of maximal D_1-asymmetric copulas fulfilling $\lim_{n\to\infty} D_\partial(A_n, A) = 0$ for some $A \in \mathcal{C}$. Then applying Theorem 4.4, the triangle inequality and the fact that D_∂-convergence implies both $\lim_{n\to\infty} \Phi_{A_n,A}\left(\frac{1}{2}\right) = 0$ and $\lim_{n\to\infty} \Phi_{A_n^t,A^t}\left(\frac{1}{2}\right) = 0$ we obtain

$$1 = \Phi_{A_n,A_n^t}\left(\tfrac{1}{2}\right) \leq \Phi_{A_n,A}\left(\tfrac{1}{2}\right) + \Phi_{A,A^t}\left(\tfrac{1}{2}\right) + \Phi_{A^t,A_n^t}\left(\tfrac{1}{2}\right)$$

and hence

$$\Phi_{A,A^t}\left(\tfrac{1}{2}\right) \geq 1 - \lim_{n\to\infty} \Phi_{A_n,A}\left(\tfrac{1}{2}\right) - \lim_{n\to\infty} \Phi_{A^t,A_n^t}\left(\tfrac{1}{2}\right) = 1.$$

□

Remark 4.7. Proposition 4.6 certainly isn't surprising, the following result, however, is. Key for proving the statement is property *(v)* of Theorem 4.4.

Theorem 4.8. *The set $\mathcal{C}^{\kappa=1}$ is closed in $(\mathcal{C}, D_1)$.*

Proof. Suppose $A, A_1, A_2, \ldots$ are copulas, that $\kappa(A_n) = 1$ for every $n \in \mathbb{N}$ and that $\lim_{n\to\infty} D_1(A_n, A) = 0$. Since the $*$-product is jointly continuous w.r.t. D_1 (see [29]) we have

$$\lim_{n\to\infty} D_1(A_n * A_n, A * A) = 0.$$

Considering that D_1-convergence implies d_∞-convergence, $\lim_{n\to\infty}(A_n * A_n)\left(\frac{1}{2},\frac{1}{2}\right) = A * A\left(\frac{1}{2},\frac{1}{2}\right)$ follows, and applying Theorem 4.4 the proof is complete. □

Analogous to the fact, that shuffles are dense in $(\mathcal{C}, d_\infty)$ the set $\mathcal{C}_{mcd}^{\kappa=1}$ is dense in $(\mathcal{C}^{\kappa=1}, d_\infty)$.

Theorem 4.9. *The set $\mathcal{C}_{mcd}^{\kappa=1}$ is dense in $(\mathcal{C}^{\kappa=1}, d_\infty)$.*

Proof. Fix $\varepsilon > 0$ and let $A \in \mathcal{C}^{\kappa=1}$ be a copula with maximal D_1-asymmetry. According to property *(iv)* in Theorem 4.4 there exist sets $U_1, U_2, V_1, V_2 \in \mathcal{B}([0,1])$ and copulas $C_1, C_2, C_3, C_4 \in \mathcal{C}$ such that

$$A(x,y) = \frac{1}{4}\Big[C_1(F_1(x), G_1(y)) + C_2(G_1(x), G_2(y)) + C_3(F_2(x), F_1(y)) + C_4(G_2(x), F_2(y))\Big],$$

whereby $F_i(x) := 4\lambda(U_i \cap [0,x]) = 4\int_{[0,x]} \mathbb{1}_{U_i}(s)\lambda(s)$ and $G_i(x) := 4\lambda(V_i \cap [0,x]) = 4\int_{[0,x]} \mathbb{1}_{V_i}(s)d\lambda(s)$ for $i \in \{1,2\}$. It is well known that $\mathcal{C}_{mcd}$ (in fact even the family of straight shuffles) is dense in $(\mathcal{C}, d_\infty)$ (see, e.g., [6][Corollary 4.1.16]), hence, we can find mutually completely dependent copulas $C_{h_1}, C_{h_2}, C_{h_3}, C_{h_4} \in \mathcal{C}_{mcd}$ with $d_\infty(C_i, C_{h_i}) < \varepsilon$ for every $i \in \{1,2,3,4\}$. Defining $\tilde{A}$ by

$$\tilde{A}(x,y) := \frac{1}{4}\Big[C_{h_1}(F_1(x), G_1(y)) + C_{h_2}(G_1(x), G_2(y)) + C_{h_3}(F_2(x), F_1(y)) + C_{h_4}(G_2(x), F_2(y))\Big],$$

and applying Theorem 4.4 we conclude that $\tilde{A}$ has maximal D_1-asymmetry too. Furthermore, using the triangle inequality we obtain

$$\begin{aligned}
\sup_{x,y\in[0,1]} |A(x,y) - \tilde{A}(x,y)| &\leq \frac{1}{4} \sup_{x,y\in[0,1]} |C_1(F_1(x), G_1(y)) - C_{h_1}(F_1(x), G_1(y))| \\
&\quad + \frac{1}{4} \sup_{x,y\in[0,1]} |C_2(G_1(x), G_2(y)) - C_{h_2}(G_1(x), G_2(y))| \\
&\quad + \frac{1}{4} \sup_{x,y\in[0,1]} |C_3(F_2(x), F_1(y)) - C_{h_3}(F_2(x), F_1(y))| \\
&\quad + \frac{1}{4} \sup_{x,y\in[0,1]} |C_4(G_2(x), F_2(y)) - C_{h_4}(G_2(x), F_2(y))| \\
&\leq \frac{1}{4} \sum_{i=1}^{4} d_\infty(C_i, C_{h_i}) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

As final step we have to show $\tilde{A} \in \mathcal{C}_{mcd}$, which can be done as follows: Fix $y \in [0,1]$, then by applying Lemma 1 in [20] using the fact that $K_{C_{h_i}}(x,[0,y])$ is given by $K_{C_{h_i}}(x,[0,y]) = \mathbb{1}_{[0,y]}(h_i(x))$ for λ-a.e. $x \in [0,1]$ and every $i \in \{1,\dots,4\}$, the Markov kernel $K_{\tilde{A}}(x,[0,y])$ of $\tilde{A}$ can be expressed by

$$K_{\tilde{A}}(x,[0,y]) = \begin{cases} \mathbb{1}_{[0,G_1(y)]}(h_1 \circ F_1(x)) & \text{for } x \in U_1 \\ \mathbb{1}_{[0,G_2(y)]}(h_2 \circ G_1(x)) & \text{for } x \in V_1 \\ \mathbb{1}_{[0,F_1(y)]}(h_3 \circ F_2(x)) & \text{for } x \in U_2 \\ \mathbb{1}_{[0,F_2(y)]}(h_4 \circ G_2(x)) & \text{for } x \in V_2 \end{cases}$$

for λ-a.e. $x \in [0,1]$. Since $\{U_1, U_2, V_1, V_2\}$ form a partition of $[0,1]$, for each $y \in [0,1]$ we have that $K_{\tilde{A}}(x,[0,y]) \in \{0,1\}$ for λ-a.e. $x \in [0,1]$, which is equivalent to $\tilde{A}$ being completely dependent (see [3]). Using the same arguments we also obtain for every $y \in [0,1]$ that $K_{\tilde{A}^t}(x,[0,y]) = (\partial_1 \tilde{A}^t)(x,y) = (\partial_2 \tilde{A})(y,x) \in \{0,1\}$ for λ-a.e. $x \in [0,1]$, i.e., $\tilde{A}^t$ is completely dependent too. Altogether, we have shown that $\tilde{A} \in \mathcal{C}_{mcd}$, which completes the proof. □

5 Maximal D_p-asymmetry

Since the metrics D_p, $p \in [1,\infty]$ induce the same topology on $\mathcal{C}$ one could conjecture that maximal D_p-asymmetry might be the same as maximal D_1-asymmetry. We will falsify this idea and start with 3 simple lemmata.

Lemma 5.1 ([27])**.** *Suppose that h_1, h_2 are λ-preserving transformations on $[0,1]$ and let A_{h_1}, A_{h_2} denote the corresponding completely dependent copulas. Then*

$$D_p^p(A_{h_1}, A_{h_2}) = D_1(A_{h_1}, A_{h_2}) = \|h_1 - h_2\|_1$$

holds for every $p \in (1, \infty)$.

Lemma 5.2. *The metric space $(\mathcal{C}, D_p)$ has the following diameter:*

1. *$diam_{D_p}(\mathcal{C}) = 2^{-\frac{1}{p}}$ for $p \in [1, \infty)$*
2. *$diam_{D_\infty}(\mathcal{C}) = 1$.*

Proof. According to Lemma 5 in [27] we have

$$diam_{D_1}(\mathcal{C}) = \int_{[0,1]} \min\{2y, 2(1-y)\} d\lambda(y) = \frac{1}{2}.$$

Since $|K_A(x, [0, y]) - K_B(x, [0, y])| \in [0, 1]$ it is straightforward to verify that

$$D_p^p(A, B) \leq D_1(A, B) \leq D_p(A, B) \tag{8}$$

holds for every $A, B \in \mathcal{C}$ and $p \in [1, \infty)$. As direct consequence we get $D_p(A, B) \leq D_1(A, B)^{\frac{1}{p}} \leq 2^{-\frac{1}{p}}$. On the other hand, there exist copulas $A, B \in \mathcal{C}$ with $D_p(A, B) = 2^{-\frac{1}{p}}$. Considering $A = M$ and $B = W$ and applying Lemma 5.1 yields

$$D_p^p(M, W) = D_1(M, W) = \frac{1}{2}.$$

The assertion for $p = \infty$ is a direct consequence of Lemma 5 in [27]. □

Slightly adapting the notation of the previous section we will now focus on the family $\mathcal{C}^{\kappa_p = 1}$ of all bivariate copulas with maximal D_p-asymmetry, i.e., $\mathcal{C}^{\kappa_p=1} := \{A \in \mathcal{C} : \kappa_p(A) := 2^{\frac{1}{p}} D_p(A, A^t) = 1\}$. Building upon Lemma 5.1 and Theorem 3.5 in [13] there are mutually completely dependent copulas $A \in \mathcal{C}_{mcd}$ such that $\kappa_p(A) = 1$ is attained for every $p \in [1, \infty]$. In fact, the copula A_h defined in Example 3.4 in [13] has maximal D_p-asymmetry for every $p \in [1, \infty]$. The following lemma shows that a copula with maximal D_p-asymmetry for $p \in (1, \infty)$ also has maximal D_1-asymmetry.

Lemma 5.3. *If $A \in \mathcal{C}$ satisfies $\kappa_p(A) = 1$ for some $p \in (1, \infty)$ then $\kappa_1(A) = 1$ holds.*

Proof. Using the inequality $D_p^p(A, B) \leq D_1(A, B)$ as well as the fact that $D_1(A, B) \leq \frac{1}{2}$ holds for all $A, B \in \mathcal{C}$ we get

$$1 = 2^{\frac{1}{p}} D_p(A, A^t) \leq \big(2\ D_1(A, A^t)\big)^{\frac{1}{p}} \leq 1,$$

which yields $D_1(A, A^t) = \frac{1}{2}$. □

The following example, however, shows that the reverse implication does not hold in general.

Example 5.4. Suppose that $A \in \mathcal{C}$ corresponds to the uniform distribution on the union of the four squares (see Figure 4)

$$\left(0, \tfrac{1}{4}\right) \times \left(\tfrac{1}{4}, \tfrac{2}{4}\right), \left(\tfrac{1}{4}, \tfrac{2}{4}\right) \times \left(\tfrac{2}{4}, \tfrac{3}{4}\right), \left(\tfrac{2}{4}, \tfrac{3}{4}\right) \times \left(\tfrac{3}{4}, 1\right), \left(\tfrac{3}{4}, 1\right) \times \left(0, \tfrac{1}{4}\right).$$

Since A (and A^t) is a checkerboard copula (see, for instance, [11, 19]) a version of the Markov kernel of A is piecewise linear in y for fixed $x \in [0,1]$ and does not depend on the choice of the point $x \in \left(\frac{i-1}{4}, \frac{i}{4}\right)$ for every $i \in \{1, \dots, 4\}$, (a version of) the corresponding Markov kernel is given by

$$K_A(x,[0,y]) = \begin{cases} (4y-1)\mathbb{1}_{\left(\frac{1}{4},\frac{2}{4}\right]}(y) + \mathbb{1}_{\left(\frac{2}{4},1\right]}(y) & \text{for } x \in \left(0,\frac{1}{4}\right) \\ (4y-2)\mathbb{1}_{\left(\frac{2}{4},\frac{3}{4}\right]}(y) + \mathbb{1}_{\left(\frac{3}{4},1\right]}(y) & \text{for } x \in \left(\frac{1}{4},\frac{2}{4}\right) \\ (4y-3)\mathbb{1}_{\left(\frac{3}{4},1\right]}(y) & \text{for } x \in \left(\frac{2}{4},\frac{3}{4}\right) \\ (4y)\mathbb{1}_{\left[0,\frac{1}{4}\right]}(y) + \mathbb{1}_{\left(\frac{1}{4},1\right]}(y) & \text{for } x \in \left(\frac{3}{4},1\right). \end{cases}$$

It is straightforward to verify $\kappa_1(A) = 1$ (e.g., by using property *(iv)* or property *(v)* in Theorem

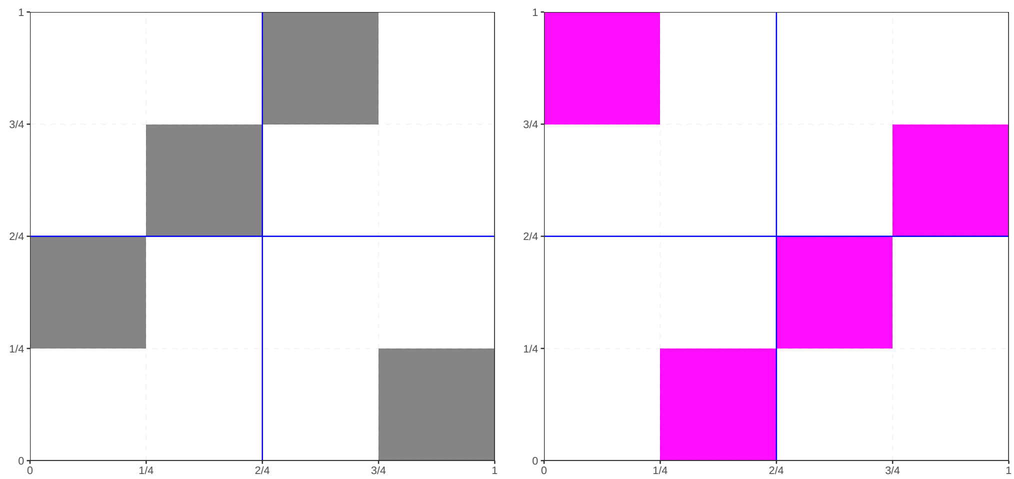

Figure 4: Density of the copula A (left panel) and the copula A^t (right panel) considered in Example 5.4. The copula A has maximal D_1-asymmetry, i.e., $\kappa_1(A) = 1$, nevertheless $\kappa_p(A) < 1$ holds for $p \in (1,\infty)$.

4.4). On the other hand, simple calculations (see Appendix 6) yield

$$D_p^p(A,A^t) = \frac{1}{4} + 2\int_{[0,\frac{1}{4}]} (4x)^p d\lambda(x) = \frac{1}{4} + \frac{2}{4p+4}$$

for every $p \in [1,\infty)$. As a direct consequence we get $D_p^p(A,A^t) < 2^{-1}$ for every $p \in (1,\infty)$, i.e., although A has maximal D_1-asymmetry, it fails to have maximal D_p-asymmetry.

Contrary to D_1, the class $\mathcal{C}^{\kappa_p=1}$, $p \in (1,\infty)$ only contains mutually completely dependent copulas.

Theorem 5.5. *If $A \in \mathcal{C}$ has maximal D_p-asymmetry for $p \in (1,\infty)$, then A is a mutually completely dependent copula.*

Proof. If $\kappa_p(A) = 1$ we have $\kappa_1(A) = 1$ and $D_p(A,A^t) = 2^{-\frac{1}{p}}$, which implies

$$\frac{1}{2} = \int_{[0,1]}\int_{[0,1]} |K_A(x,[0,y]) - K_{A^t}(x,[0,y])|^p \, d\lambda(x)d\lambda(y)$$

$$\leq \int_{[0,1]} \int_{[0,1]} |K_A(x,[0,y]) - K_{A^t}(x,[0,y])| \, d\lambda(x) d\lambda(y) = \frac{1}{2}.$$

Therefore we obtain

$$|K_A(x,[0,y]) - K_{A^t}(x,[0,y])|^p = |K_A(x,[0,y]) - K_{A^t}(x,[0,y])| \,,$$

or equivalently, that

$$|K_A(x,[0,y]) - K_{A^t}(x,[0,y])| \in \{0,1\} \tag{9}$$

holds for λ^2-a.e. $(x,y) \in [0,1]^2$. According to Lemma 5.3 and property *(iii)* in Theorem 4.4 there exist sets $U \in \mathcal{B}([0,1])$ and $V \in \mathcal{B}([0,1])$ such that $U \cap V = \emptyset$, $\lambda(U) = \lambda(V) = \frac{1}{2}$ and

$$K_A(x,[0,y]) = \begin{cases} \leq 1 & \text{for every } y \in \left[0,\frac{1}{2}\right] \\ 1 & \text{for every } y \in \left(\frac{1}{2},1\right] \end{cases} \qquad K_{A^t}(x,[0,y]) = \begin{cases} 0 & \text{for every } y \in \left[0,\frac{1}{2}\right] \\ \leq 1 & \text{for every } y \in \left(\frac{1}{2},1\right] \end{cases},$$

for every $x \in U$ as well as

$$K_A(x,[0,y]) = \begin{cases} 0 & \text{for every } y \in \left[0,\frac{1}{2}\right] \\ \leq 1 & \text{for every } y \in \left(\frac{1}{2},1\right] \end{cases} \qquad K_{A^t}(x,[0,y]) = \begin{cases} \leq 1 & \text{for every } y \in \left[0,\frac{1}{2}\right] \\ 1 & \text{for every } y \in \left(\frac{1}{2},1\right] \end{cases},$$

for every $x \in V$. Fix $x \in U$ such that Eq. (9) holds and suppose that $K_A(x,[0,y]) = y_0 \in (0,1)$ for some $y \in \left[0,\frac{1}{2}\right]$. Then due to Eq. (9) the Markov kernel of A^t must satisfy $K_{A^t}(x,[0,y]) = y_0$, which is a contradiction to the fact that $K_{A^t}(x,[0,y]) = 0$ for every $y \in \left[0,\frac{1}{2}\right]$. Hence we obtain that $K_A(x,[0,y]) \in \{0,1\}$ for every $y \in [0,1]$. In an analogous way we obtain that $K_{A^t}(x,[0,y]) \in \{0,1\}$ holds for every $y \in [0,1]$. Proceeding in the exactly same manner for $x \in V$ we get that for λ-a.e. $x \in [0,1]$ and every $y \in [0,1]$ the Markov kernels of A and A^t satisfy $K_A(x,[0,y]) \in \{0,1\}$ and $K_{A^t}(x,[0,y]) \in \{0,1\}$. By Theorem 7.1 in [3] and Lemma 3.4 in [10] it follows that A and A^t are completely dependent, implying that A is mutually completely dependent. □

Altogether we have shown the following results:

Corollary 5.6. *The following properties hold:*

1. $\mathcal{C}^{\kappa_1=1} = \mathcal{C}^{\kappa_\infty=1}$.
2. $\mathcal{C}^{\kappa_1=1} \supsetneq \mathcal{C}^{\kappa_p=1}$ *for every* $p \in (1,\infty)$.
3. $\mathcal{C}^{\kappa_1=1}_{mcd} = \mathcal{C}^{\kappa_p=1}$ *for every* $p \in (1,\infty)$.

6 Maximal D_p-asymmetric copulas and their values for ζ_1 and ξ

In Section 3 we have shown that copulas $A \in \mathcal{C}$ with maximal d_∞-asymmetry have very high dependence scores with respect to ζ_1 and ξ. Here we now focus on the range of these dependence measures to maximal D_p-asymmetric copulas.

Theorem 6.1. *If $A \in \mathcal{C}$ satisfies $\kappa_p(A) = 1$ for some $p \in (1,\infty)$ then $\zeta_1(A) = \xi(A) = 1$.*

Proof. Since $\zeta_1(A)$ and $\xi(A)$ are 1 if and only if A is completely dependent, the assertion directly follows from Theorem 5.5. □

For the case $p=1$ different values for ξ and ζ_1 are possible.

Theorem 6.2. *If $A \in \mathcal{C}$ satisfies $\kappa_1(A) = 1$ then $\xi(A) \in \left(\frac{1}{2}, 1\right]$ holds.*

Proof. Proceeding analogously to the proof of Theorem 4.4 we obtain $(A^t * A)\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$ if $\kappa_1(A) = 1$ (see Appendix 6). Since $A^t * A$ is a copula, we find copulas $A_1, A_2 \in \mathcal{C}$ with $(A^t * A) = \left(\langle \frac{i-1}{2}, \frac{i}{2}, A_i \rangle\right)_{i \in \{1,2\}} =: \tilde{A}$. Setting $C_\Pi := \left(\langle \frac{i-1}{2}, \frac{i}{2}, \Pi \rangle\right)_{i \in \{1,2\}}$, $\mu_{A^t * A} \neq \mu_{C_\Pi}$ follows. In fact, according to Theorem 4.4 there exists a set $U \in \mathcal{B}([0,1])$ such that $\lambda\left(U \cap \left[0, \frac{1}{2}\right]\right) = \lambda(U \cap \left[\frac{1}{2}, 1\right]) = \frac{1}{4}$ and $0 = \mu_A([0, \frac{1}{2}] \times U) = \int_{[0,\frac{1}{2}]} K_A(x, U) d\lambda(x)$. Hence, we can find Borel sets $\Lambda_1 \subseteq [0, \frac{1}{2}]$, $\Lambda_2 \subseteq (\frac{1}{2}, 1]$ such that $\lambda(\Lambda_1) = \lambda(\Lambda_2) = \frac{1}{2}$ and $K_A(x, U) = 0$ for all $x \in \Lambda_1$ and $K_A(x, U) = 1$ for all $x \in \Lambda_2$. The set Λ_3 defined by $[0,1] \setminus (\Lambda_1 \cup \Lambda_2)$ obviously fulfils $\lambda(\Lambda_3) = 0$. Hence, we have

$$\begin{aligned}
\mu_{A^t * A}(U \times U) &= \int_U \int_{[0,1]} K_A(s, U) K_{A^t}(x, ds) d\lambda(x) \\
&= \int_U K_{A^t}(x, \Lambda_2) d\lambda(x) + \int_U \int_{\Lambda_3} K_A(s, U) K_{A^t}(x, ds) d\lambda(x) \\
&\geq \int_U K_{A^t}(x, \Lambda_2) d\lambda(x) = \mu_{A^t}(U \times \Lambda_2) = \mu_A(\Lambda_2 \times U) = \int_{\Lambda_2} K_A(x, U) d\lambda(x) \\
&= \int_{\left(\frac{1}{2}, 1\right]} K_A(x, U) d\lambda(x) = \mu_A\left(\left(\tfrac{1}{2}, 1\right] \times U\right) = \lambda(U) - \mu_A\left(\left[0, \tfrac{1}{2}\right] \times U\right) \\
&= \lambda(U) = \frac{1}{2}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\mu_{C_\Pi}(U \times U) &= \int_U K_{C_\Pi}(x, U) d\lambda(x) = \int_{U \cap [0, \frac{1}{2}]} K_{C_\Pi}(x, U) d\lambda(x) + \int_{U \cap (\frac{1}{2}, 1]} K_{C_\Pi}(x, U) d\lambda(x) \\
&= \int_{U \cap [0, \frac{1}{2}]} 2\lambda(U \cap [0, \tfrac{1}{2}]) d\lambda(x) + \int_{U \cap (\frac{1}{2}, 1]} 2\lambda(U \cap \left(\tfrac{1}{2}, 1\right]) d\lambda(x) = \frac{1}{4}
\end{aligned}$$

holds, implying $A_i \neq \Pi$ for $i = 1, 2$, hence, considering that according to Lemma 4.3 we have $D_2^2(A, \Pi) \geq D_2^2(A^t * A, A^t * \Pi) = D_2^2(A^t * A, \Pi)$ and applying Corollary 3.2 finally yields

$$1 \geq \xi(A) \geq \xi(\tilde{A}) = \frac{1}{4} \xi(A_1) + \frac{1}{4} \xi(A_2) + \frac{1}{2} > \frac{1}{2}.$$

□

Theorem 6.3. *If $A \in \mathcal{C}$ satisfies $\kappa_1(A) = 1$, then $\zeta_1(A) \in \left[\frac{3}{4}, 1\right]$ holds.*

Proof. Using the same arguments as in the proof of Theorem 6.2 we find copulas $A_1, A_2 \in \mathcal{C}$ such that $(A^t * A) = \left(\langle \frac{i-1}{2}, \frac{i}{2}, A_i \rangle\right)_{i \in \{1,2\}} =: \tilde{A}$ holds. Since $\tilde{A}$ is an ordinal sum it is clear that the (SI)-rearrangement $\tilde{A}^\uparrow$ of $\tilde{A}$ satisfies $\tilde{A}^\uparrow = \left(\left\langle \frac{i-1}{2}, \frac{i}{2}, A_i^\uparrow \right\rangle\right)_{i \in \{1,2\}}$. As stochastically increasing copula, $A_i^\uparrow$ fulfills $A_i^\uparrow(x, y) \geq \Pi(x, y)$ for all $(x, y) \in [0, 1]^2$ and $i \in \{1, 2\}$, implying $\tilde{A}^\uparrow(x, y) \geq C_\Pi(x, y)$ for every $(x, y) \in [0, 1]^2$, whereby C_Π is defined as $C_\Pi := \left(\langle \frac{i-1}{2}, \frac{i}{2}, \Pi \rangle\right)_{i \in \{1,2\}}$. Hence, using Lemma 4.3 we get

$$\frac{1}{3} \geq D_1(A, \Pi) \geq D_1(A^t * A, A^t * \Pi) = D_1(A^t * A, \Pi) = D_1(\tilde{A}^\uparrow, \Pi) \geq D_1(C_\Pi, \Pi),$$

whereby we used the fact that $D_1(A,\Pi)$ is monotone with respect to the pointwise order in $\mathcal{C}^{\uparrow}$ (see [26]). Using Lemma 3.1 we obtain

$$\begin{aligned}\zeta_1(C_\Pi) &= 3D_1(C_\Pi,\Pi)\\ &= \frac{3}{4}\int_{[0,1]}\int_{[0,1]}\left|y-\frac{y}{2}\right| d\lambda(x)d\lambda(y) + \frac{3}{4}\int_{[0,1]}\int_{[0,1]}\left|y-\frac{1}{2}-\frac{y}{2}\right| d\lambda(x)d\lambda(y) + \frac{3}{8}\\ &= \frac{3}{16}+\frac{3}{8}-\frac{3}{16}+\frac{3}{8}=\frac{6}{8}=\frac{3}{4},\end{aligned}$$

which completes the proof. ☐

The following example demonstrates that it is possible to find copulas $A \in \mathcal{C}^{\kappa_1=1}$ such that $\zeta_1(A)$ (or $\xi(A)$, respectively) is arbitrarily close to the lower bound derived in Theorem 6.2 and Theorem 6.3.

Example 6.4. Let $n \in \mathbb{N}$ be a natural number with $n \geq 3$, set $N := 2^n$ and define the sets $U_N^1, U_N^2, V_N^1, V_N^2 \in \mathcal{B}([0,1])$ by

$$U_N^1 = \bigcup_{j=1}^{\frac{N}{4}}\left(\frac{2j-2}{N},\frac{2j-1}{N}\right), \qquad V_N^1 = \bigcup_{j=1}^{\frac{N}{4}}\left(\frac{2j-1}{N},\frac{2j}{N}\right),$$
$$U_N^2 = \bigcup_{j=1}^{\frac{N}{4}}\left(\frac{1}{2}+\frac{2j-2}{N},\frac{1}{2}+\frac{2j-1}{N}\right), V_N^2 = \bigcup_{j=1}^{\frac{N}{4}}\left(\frac{1}{2}+\frac{2j-2}{N},\frac{1}{2}+\frac{2j-1}{N}\right).$$

Obviously, we have $\lambda(U_N^1) = \lambda(U_N^2) = \lambda(V_N^1) = \lambda(V_N^2) = \frac{1}{4}$ and $\lambda(U_N^1 \cup U_N^2 \cup V_N^1 \cup V_N^2) = 1$. Letting A_N denoting the copula corresponding to the uniform distribution on the union of the four sets $U_N^1 \times V_N^1$, $U_N^2 \times U_N^1$, $V_N^1 \times V_N^2$ and $V_N^2 \times U_N^2$ (see Figure 5), then by Theorem 4.4 A_N has maximal D_1-asymmetry. As next step we calculate the dependence measure $\zeta_1(A_N)$.
Applying Lemma 6.3 in [10] and using the fact that for every $y \in [0,1]$ the identity of $K_{A_N}(x_1,[0,y]) = K_{A_N}(x_2,[0,y])$ holds for λ-a.e. $x_1, x_2 \in X$, whereby $X \in \{U_N^1, U_N^2, V_N^1, V_N^2\}$, we obtain

$$\begin{aligned}\frac{\zeta_1(A_N)}{3} &= D_1(A_N,\Pi) \leq \frac{2}{N} + \frac{1}{N}\sum_{j=1}^{N}\int_{[0,1]}\left|K_{A_N}\left(x,\left[0,\frac{j}{N}\right]\right)-\frac{j}{N}\right| d\lambda(x)\\ &= \frac{2}{N} + \sum_{X\in\{U_N^1,U_N^2,V_N^1,V_N^2\}}\underbrace{\frac{1}{N}\sum_{j=1}^{N}\int_X\left|K_{A_N}\left(x,\left[0,\frac{j}{N}\right]\right)-\frac{j}{N}\right| d\lambda(x)}_{=:m(X)}.\end{aligned}$$

Considering $X = U_N^1$ and $x \in U_N^1$, a version of the Markov kernel $K_{A_n}\left(x,\left[0,\frac{j}{N}\right]\right)$ is given by

$$K_{A_N}\left(x,\left[0,\frac{j}{N}\right]\right) = \begin{cases}\frac{2j}{N} & \text{for } j \in \{2,4,\ldots,\frac{N}{2}\}\\ \frac{2(j-1)}{N} & \text{for } j \in \{1,3,\ldots,\frac{N}{2}-1\}\\ 1 & \text{for } j > \frac{N}{2},\end{cases}$$

which yields

$$m(U_N^1) = \frac{1}{4N}\sum_{j=1}^{N}\left|K_{A_N}\left(x,\left[0,\frac{j}{N}\right]\right)-\frac{j}{N}\right|$$

$$= \frac{1}{4N}\left(\sum_{j\in\{2,4,\dots,N/2\}}\left|\frac{2j}{N}-\frac{j}{N}\right| + \sum_{j\in\{1,3,\dots,N/2-1\}}\left|\frac{2(j-1)}{N}-\frac{j}{N}\right| + \sum_{j=\frac{N}{2}+1}^{N}\left(1-\frac{j}{N}\right)\right)$$
$$= \frac{1}{4N}\left(\sum_{j=1}^{\frac{N}{4}}\left|\frac{4j}{N}-\frac{2j}{N}\right| + \sum_{j=1}^{\frac{N}{4}}\left|\frac{4(j-1)}{N}-\frac{2j-1}{N}\right| + \sum_{j=1}^{\frac{N}{2}}\left(\frac{1}{2}-\frac{j}{N}\right)\right)$$
$$= \frac{1}{4N}\left(\frac{N}{4}+\frac{2}{N}-\frac{1}{2}\right) \le \frac{1}{16}+\frac{1}{2N^2}.$$

In a similar manner we obtain $m(U_N^2) = \frac{1}{16}+\frac{1}{8N}$, $m(V_N^1) = \frac{1}{16}+\frac{1}{8N}$ and $m(V_N^2) \le \frac{1}{16}+\frac{1}{2N^2}$. Together with Theorem 6.3 it follows that

$$\frac{3}{4} \le \zeta_1(A_N) \le \frac{3}{4}+\frac{27}{4N}+\frac{3}{N^2},$$

which shows that for sufficiently large $N \in \mathbb{N}$ the dependence value $\zeta_1(A_N)$ is arbitrarily close to $\frac{3}{4}$.
Using similar calculations (see Appendix 6) yields

$$\frac{1}{2} < \xi(A_N) \le \frac{1}{2} + a_N,$$

whereby $\lim_{N\to\infty} a_N = 0$.

Remark 6.5. Slightly modifying the construction from Example 6.4 (which corresponds to copying shrunk versions of the product copula Π in the small squares) we now construct the copula B_N by copying shrunk versions of M in every square of the 'diagonal' of each of the four sets $U_N^1 \times V_N^1$, $U_N^2 \times U_N^1$, $V_N^1 \times V_N^2$ and $V_N^2 \times U_N^2$ as depicted in Figure 5 (magenta lines). The shuffle B_N is obviously maximal D_1-asymmetric and, being completely dependent, fulfills $\zeta_1(B_N) = 1 = \xi(B_N)$.

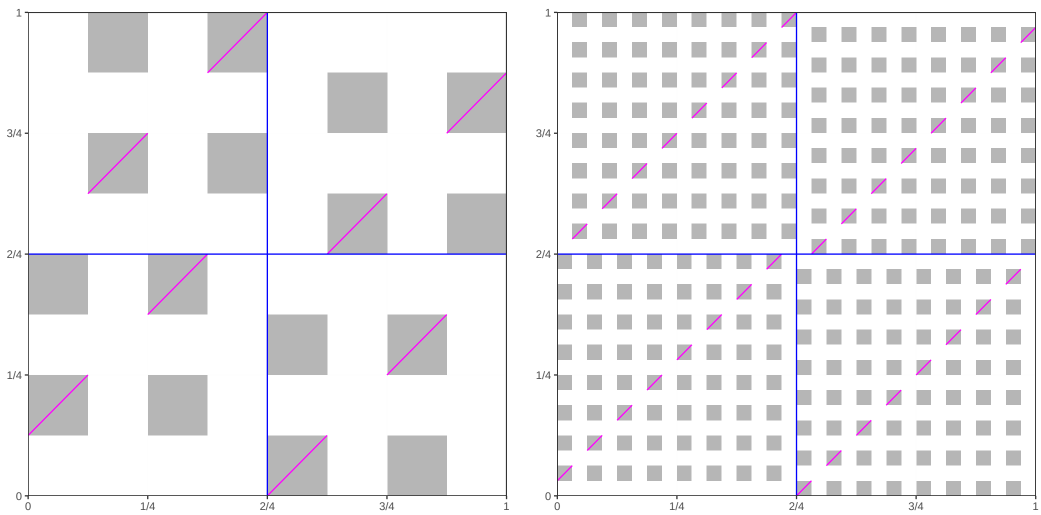

Figure 5: Density of the copula A_N (gray) as considered in Example 6.4 and support of the mutually completely dependent copula B_N (magenta) according to Remark 6.5 for $N = 8$ (left panel) and $N = 32$ (right panel).

Hence, setting $C_N^\alpha := \alpha A_N + (1-\alpha)B_N$ for every $\alpha \in [0,1]$ (with A_N according to Example 6.4) obviously yields a maximal D_1-asymmetric copula C_N^α. Due to the fact that $\zeta_1(C_N^\alpha)$ and $\xi(C_N^\alpha)$ is continuous in α the intermediate value theorem implies that that for every $s \in [\zeta_1(A_N), 1]$ we can find a copula C_N^α with $\zeta_1(C_N^\alpha) = s$ and the same result holds for ζ_1 replaced by ξ. In other words, each point in the intervals mentioned in Theorem 6.2 and Theorem 6.3 is attained.

Remark 6.6. We have neither been able to find a copula $A \in \mathcal{C}^{\kappa_1=1}$ fulfilling $\zeta_1(A) = \frac{3}{4}$, nor to prove that such a copula does not exist.

Acknowledgements
The authors gratefully acknowledge the support of the WISS 2025 project 'IDA-lab Salzburg' (20204-WISS/225/197-2019 and 20102-F1901166-KZP)

Appendix

Calculations for the proof of Theorem 3.5:
To calculate $\zeta_1(C_\Pi)$ we apply Lemma 3.1 and obtain

$$\begin{aligned}\zeta_1(C_\Pi) = 3D_1(C_\Pi, \Pi) &= \frac{1}{3}\int_0^1\int_0^1 \left|y - \frac{y}{3}\right| d\lambda(x)d\lambda(y) + \frac{1}{3}\int_0^1\int_0^1 \left|y - \frac{1}{3} - \frac{y}{3}\right| d\lambda(x)d\lambda(y) + \\ &+ \frac{1}{3}\int_0^1\int_0^1 \left|y - \frac{2}{3} - \frac{y}{3}\right| d\lambda(x)d\lambda(y) + \frac{10}{18} \\ &= \frac{1}{3}\int_{[0,1]} \frac{2y}{3} d\lambda(y) + \frac{1}{3}\int_{[0,1]} \left|\frac{2y-1}{3}\right| d\lambda(y) + \frac{1}{3}\int_{[0,1]} \frac{2-2y}{3} d\lambda(y) + \frac{10}{18} \\ &= \frac{2}{18} + \frac{1}{18} + \frac{2}{18} + \frac{10}{18} = \frac{15}{18} = \frac{5}{6}.\end{aligned}$$

Let A_s be the ordinal sum of C_s, i.e., $A_s := \left(\langle \frac{k-1}{3}, \frac{k}{3}, C_s\rangle\right)_{k\in\{1,2,3\}}$, whereby C_s is the ordinal sum considered in Example 3.3. Then for the integrals considered in Lemma 3.1 we obtain for $s \in [0, \frac{1}{3}]$

$$\begin{aligned}I_1 &= \int_{[0,1]}\int_{[0,1]} |y - ys|\, dxdy = \frac{1-s}{2} \\ I_2 &= \int_{[0,1]}\int_{[0,1]} \left|\mathbb{1}_{[0,y]}(x) - (s + y(\tfrac{1}{3} - s))\right| dxdy \\ &= \int_{[0,1]} y(1 - (s + y(\tfrac{1}{3} - s)))dy + \int_{[0,1]} (1-y)(s + y(\tfrac{1}{3} - s))dy \\ &= \frac{1}{18}(7 - 3s) + \frac{1}{18}(6s + 1) = \frac{1}{18}(3s + 8) \\ I_3 &= \int_{[0,1]}\int_{[0,1]} \left|y - (\tfrac{1}{3} + sy)\right| = \frac{9s^2 - 12s + 5}{18 - 18s} \\ I_4 &= \int_{[0,1]}\int_{[0,1]} \left|\mathbb{1}_{[0,y]}(x) - (\tfrac{1}{3} + s + y(\tfrac{1}{3} - s))\right| dxdy = \frac{1}{18}(4 - 3s) + \frac{1}{18}(6s + 4) = \frac{1}{18}(3s + 8) \\ I_5 &= \int_{[0,1]}\int_{[0,1]} \left|y - (\tfrac{2}{3} + sy)\right| = \frac{9s^2 - 6s + 5}{18 - 18s} \\ I_6 &= \int_{[0,1]}\int_{[0,1]} \left|\mathbb{1}_{[0,y]}(x) - (\tfrac{2}{3} + s + y(\tfrac{1}{3} - s))\right| dxdy = \frac{1}{18}(1 - 3s) + \frac{1}{18}(6s + 7) = \frac{1}{18}(3s + 8).\end{aligned}$$

Applying Lemma 3.1 again we get

$$\begin{aligned}D_1(A_s, \Pi) &= s^2 I_1 + \left(\frac{1}{3} - s\right)^2 I_2 + s^2 I_3 + \left(\frac{1}{3} - s\right)^2 I_4 + s^2 I_5 + \left(\frac{1}{3} - s\right)^2 I_6 + \frac{5}{54}(2 + 9s - 27s^2) \\ &= \frac{9s^4 - 4s^2 - 3s + 3}{9 - 9s} \in \left[\frac{5}{18}, \frac{1}{3}\right]\end{aligned}$$

for $s \in \left[0, \frac{1}{3}\right]$. Since $f(s) := D_1(A_s, \Pi)$ is a continuous and decreasing function on $s \in \left[0, \frac{1}{3}\right]$ and $f(0) = \frac{1}{3}$ and $f(\frac{1}{3}) = \frac{5}{18}$, we have shown that $\zeta_1(A_s)$ attains every value in $\left[\frac{5}{6}, 1\right]$.

Calculations for Example 5.4: Partitioning the integration area and using the fact that (a version of) the Markov kernel of A and A^t does not depend on the choice of the point $x \in \left(\frac{i-1}{4}, \frac{i}{4}\right)$ we obtain

$$\begin{aligned}
D_p^p(A, A^t) &= \int_{[0,1]} \int_{[0,1]} |K_A(x, [0, y]) - K_{A^t}(x, [0, y])|^p d\lambda(x) d\lambda(y) \\
&= \frac{2}{4} \left(\int_{[\frac{1}{4}, \frac{2}{4}]} (4y - 1)^p d\lambda(y) + \int_{[\frac{2}{4}, \frac{3}{4}]} 1 d\lambda(y) + \int_{[\frac{3}{4}, \frac{4}{4}]} (1 - 4y + 3)^p d\lambda(y) \right) \\
&\quad + \frac{2}{4} \left(\int_{[\frac{0}{4}, \frac{1}{4}]} (4y)^p d\lambda(y) + \int_{[\frac{1}{4}, \frac{2}{4}]} 1 d\lambda(y) + \int_{[\frac{2}{4}, \frac{3}{4}]} (1 - 4y + 2)^p d\lambda(y) \right) \\
&= \int_{[0, \frac{1}{4}]} (4x)^p d\lambda(x) + \frac{1}{4} + \int_{[0, \frac{1}{4}]} (4x)^p d\lambda(x) \\
&= \frac{1}{4} + 2 \int_{[0, \frac{1}{4}]} (4x)^p d\lambda(x).
\end{aligned}$$

Calculations for the proof of Theorem 6.2: To show that $A^t * A(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$ follows from $\kappa_1(A) = 1$ we can proceed as follows: According to Theorem 4.4 property *(iii)* and using disintegration there exists a Borel set $U \in \mathcal{B}([0, 1])$ with $\lambda(U) = \frac{1}{2}$ and $K_A(x, [0, \frac{1}{2}]) = 1$ as well as $K_{A^t}(x, [0, \frac{1}{2}]) = 0$ for every $x \in U$. Using disintegration and Eq. (2) again yields the existence of a Borel set $V \subseteq U^c$ with $\lambda(V) = \frac{1}{2}$ and $K_A(x, [0, \frac{1}{2}]) = 0$ and $K_{A^t}(x, [0, \frac{1}{2}]) = 1$ for every $x \in V$. Set $\tilde{V} := U^c \setminus V$, then applying Lemma 2.2 yields

$$\begin{aligned}
A^t * A\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \mu_{A^t * A}\left(\left[0, \tfrac{1}{2}\right] \times \left[0, \tfrac{1}{2}\right]\right) &= \int_{[0, \frac{1}{2}]} \int_{[0,1]} K_A\left(s, \left[0, \tfrac{1}{2}\right]\right) K_{A^t}(x, ds) d\lambda(x) \\
&\leq \int_{[0, \frac{1}{2}]} \int_U 1\, K_{A^t}(x, ds) d\lambda(x) + \int_{[0, \frac{1}{2}]} \int_{\tilde{V}} 1\, K_{A^t}(x, ds) d\lambda(x) \\
&= \mu_{A^t}\left(\left[0, \tfrac{1}{2}\right] \times U\right) + \mu_{A^t}\left(\left[0, \tfrac{1}{2}\right] \times \tilde{V}\right) \leq \mu_A\left(U \times \left[0, \tfrac{1}{2}\right]\right) + \lambda(\tilde{V}) = \frac{1}{2},
\end{aligned}$$

as well as

$$\begin{aligned}
\mu_{A^t * A}\left(\left[0, \tfrac{1}{2}\right] \times \left[0, \tfrac{1}{2}\right]\right) &= \int_{[0, \frac{1}{2}]} \left(\int_U 1\, K_{A^t}(x, ds) + \int_V 0\, K_{A^t}(x, ds) + \int_{\tilde{V}} K_A\left(s, \left[0, \tfrac{1}{2}\right]\right) K_{A^t}(x, ds) \right) d\lambda(x) \\
&\geq \int_{[0, \frac{1}{2}]} K_{A^t}(x, U) d\lambda(x) = \mu_{A^t}\left(\left[0, \tfrac{1}{2}\right] \times U\right) = \mu_A\left(U \times \left[0, \tfrac{1}{2}\right]\right) = \frac{1}{2}.
\end{aligned}$$

Altogether we have shown $A^t * A(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$.

Additional calculations for Example 6.4: First of all we derive a result similar to that of Lemma 6.3 in [10] for $D_2^2(A, \Pi)$. Using Eq. (2) we obtain

$$\begin{aligned}
&\left| \int_{[0,1]} \int_{[0,1]} K_A(x, [0, y])^2 d\lambda(x) d\lambda(y) - \int_{[0,1]} \frac{1}{n} \sum_{i=1}^{n} K_A(x, [0, \tfrac{i}{n}])^2 d\lambda(x) \right| \\
&\qquad \leq \sum_{i=1}^{n} \int_{[0,1]} \int_{[\frac{i-1}{n}, \frac{i}{n}]} |K_A(x, [0, y])^2 - K_A(x, [0, \tfrac{i}{n}])^2| d\lambda(y) d\lambda(x)
\end{aligned}$$

$$\leq \frac{1}{n}\sum_{i=1}^{n}\int_{[0,1]} K_A(x,[0,\tfrac{i}{n}])^2 - K_A(x,[0,\tfrac{i-1}{n}])^2 d\lambda(x)$$
$$= \frac{1}{n}\sum_{i=1}^{n}\int_{[0,1]} \left(K_A(x,[0,\tfrac{i}{n}]) - K_A(x,[0,\tfrac{i-1}{n}])\right)\left(K_A(x,[0,\tfrac{i}{n}]) + K_A(x,[0,\tfrac{i-1}{n}])\right) d\lambda(x)$$
$$\leq \frac{2}{n}\sum_{i=1}^{n}\int_{[0,1]} K_A(x,(\tfrac{i-1}{n},\tfrac{i}{n}])d\lambda(x) = \frac{2}{n}\sum_{i=1}^{n}\lambda\left(\left(\frac{i-1}{n},\frac{i}{n}\right]\right) = \frac{2}{n}.$$

Proceeding analogously to Example 6.4 and applying the previous inequality yields

$$\frac{\xi(A_N)+2}{6} = \frac{6D_2^2(A_N,\Pi)+2}{6} = D_2^2(A_N,\Pi) + \frac{1}{3} = \int_{[0,1]}\int_{[0,1]} K_{A_N}(x,[0,y])^2 d\lambda(x)d\lambda(y) - \frac{1}{3} + \frac{1}{3}$$
$$= \int_{[0,1]}\int_{[0,1]} K_{A_N}(x,[0,y])^2 d\lambda(x)d\lambda(y) \leq \frac{2}{N} + \frac{1}{N}\sum_{j=1}^{N}\int_{[0,1]} K_{A_N}\left(x,\left[0,\frac{j}{N}\right]\right)^2 d\lambda(x)$$
$$= \frac{2}{N} + \sum_{X\in\{U_N^1,U_N^2,V_N^1,V_N^2\}} \underbrace{\frac{1}{N}\sum_{j=1}^{N}\int_X K_{A_N}\left(x,\left[0,\frac{j}{N}\right]\right)^2 d\lambda(x)}_{=:m(X)}.$$

Considering $X = U_N^1$ and $x \in U_N^1$ we obtain

$$m(U_N^1) = \frac{1}{4N}\sum_{j=1}^{N} K_{A_N}\left(x,\left[0,\frac{j}{N}\right]\right)^2 = \frac{1}{4N}\left(\sum_{j=1}^{\frac{N}{4}}\left(\frac{4j}{N}\right)^2 + \sum_{j=1}^{\frac{N}{4}}\left(\frac{4(j-1)}{N}\right)^2 + \frac{N}{2}\right)$$
$$= \frac{1}{4N}\left(\frac{(N+2)(N+4)}{12N} + \frac{N^2-6N+8}{12N} + \frac{N}{2}\right) = \frac{1}{6} + \frac{1}{3N^2}.$$

In an analogous manner we get

$$m(U_N^2) = \frac{1}{24} + \frac{1}{4N} + \frac{1}{3N^2},\; m(V_N^1) = \frac{1}{24} + \frac{1}{3N^2},\; m(V_N^2) = \frac{1}{6} + \frac{1}{4N} + \frac{1}{3N^2},$$

which altogether yields $\frac{1}{2} < \xi(A_N) \leq \frac{1}{2} + \frac{15}{N} + \frac{8}{N^2}$.

References

[1] M. Azadkia and S. Chatterjee. A simple measure of conditional dependence. *The Annals of Statistics*, 49(6):3070 – 3102, 2021.

[2] S. Chatterjee. A new coefficient of correlation. *Journal of the American Statistical Association*, 116(536):2009–2022, 2021.

[3] W. F. Darsow, B. Nguyen, and E. T. Olsen. Copulas and markov processes. *Illinois Journal of Mathematics*, 36(4):600 – 642, 1992.

[4] H. Dette, K. F. Siburg, and P. A. Stoimenov. A copula-based non-parametric measure of regression dependence. *Scandinavian Journal of Statistics*, 40(1):21–41, 2013.

[5] F. Durante, P. Sarkoci, and C. Sempi. Shuffles of copulas. *Journal of Mathematical Analysis and Applications*, 352(2):914–921, 2009.

[6] F. Durante and C. Sempi. *Principles of copula theory.* CRC Press, Hoboken, NJ, 2015.

[7] J. Fernández-Sánchez and W. Trutschnig. Conditioning-based metrics on the space of multivariate copulas and their interrelation with uniform and levelwise convergence and iterated function systems. *Journal of Theoretical Probability*, 28:1311–1336, 11 2015.

[8] J. Galambos. *Exchangeability.* John Wiley & Sons, Ltd, 2014.

[9] F. Griessenberger, J. Fernández Sánchez, and W. Trutschnig. Some properties of double shuffles of bivariate copulas and (extreme) copulas invariant with respect to łüroth double shuffles. *Fuzzy Sets and Systems*, 428:102–120, 2022.

[10] F. Griessenberger, R. R. Junker, and W. Trutschnig. On a multivariate copula-based dependence measure and its estimation. *Electronic Journal of Statistics*, 16(1):2206 – 2251, 2022.

[11] R. R. Junker, F. Griessenberger, and W. Trutschnig. Estimating scale-invariant directed dependence of bivariate distributions. *Computational Statistics & Data Analysis*, 153:107058, 2021.

[12] O. Kallenberg. *Foundations of modern probability.* Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.

[13] N. Kamnitui, J. Fernández-Sánchez, and W. Trutschnig. Maximum asymmetry of copulas revisited. *Dependence Modeling*, 6(1):47–62, 2018.

[14] T. M. Kasper, S. Fuchs, and W. Trutschnig. On weak conditional convergence of bivariate Archimedean and extreme value copulas, and consequences to nonparametric estimation. *Bernoulli*, 27(4):2217 – 2240, 2021.

[15] E. P. Klement and R. Mesiar. How non-symmetric can a copula be? *Commentationes Mathematicae Universitatis Carolinae*, 47, 01 2006.

[16] A. Klenke. *Wahrscheinlichkeitstheorie.* Springer-Lehrbuch Masterclass Series. Springer, Berlin Heidelberg, 2008.

[17] D. Kokol Bukovšek, T. Košir, B. Mojškerc, and M. Omladič. Relation between non-exchangeability and measures of concordance of copulas. *Journal of Mathematical Analysis and Applications*, 487(1):123951, 2020.

[18] P. Mikusinski and M. D. Taylor. Markov operators and n-copulas. *Annales Polonici Mathematici*, 96:75–95, 2009.

[19] P. Mikusiński and M. Taylor. Some approximations of n-copulas. *Metrika*, 72:385–414, 11 2010.

[20] T. Mroz, J. F. Sánchez, S. Fuchs, and W. Trutschnig. On distributions with fixed marginals maximizing the joint or the prior default probability, estimation, and related results. *arXiv:1602.05807v2*, 2021.

[21] R. Nelsen. Extremes of nonexchangeability. *Statistical Papers*, 48:329–336, 2007.

[22] R. B. Nelsen. *An Introduction to Copulas (Springer Series in Statistics).* Springer-Verlag, Berlin, Heidelberg, 2006.

[23] J. V. Ryff. Measure preserving transformations and rearrangements. *Journal of Mathematical Analysis and Applications*, 31:449–458, 1970.

[24] H. Shi, M. Drton, and F. Han. On the power of Chatterjee's rank correlation. *Biometrika*, 10 2021. asab028.

[25] K. F. Siburg and P. A. Stoimenov. Symmetry of functions and exchangeability of random variables. *Statistical Papers*, 52:1–15, 2011.

[26] C. Strothmann, H. Dette, and K. F. Siburg. Rearranged dependence measures. *arXiv:2201.03329v2*, 2022.

[27] W. Trutschnig. On a strong metric on the space of copulas and its induced dependence measure. *Journal of Mathematical Analysis and Applications*, 384(2):690 – 705, 2011.

[28] W. Trutschnig. Some smoothing properties of the star product of copulas. In R. Kruse, M. R. Berthold, C. Moewes, M. Á. Gil, P. Grzegorzewski, and O. Hryniewicz, editors, *Synergies of Soft Computing and Statistics for Intelligent Data Analysis*, pages 349–357, Berlin, Heidelberg, 2013. Springer Berlin Heidelberg.

[29] W. Trutschnig and J. Fernández Sánchez. Idempotent and multivariate copulas with fractal support. *Journal of Statistical Planning and Inference*, 142(12):3086–3096, 2012.